**MAT 254 – Winter Quarter 2003**
**Final Exam – Answers**

**NAME____________________________________________________**

**Show work and write clearly. Answers without work to support them will not receive full credit. Answers without correct notation will not receive full credit.**

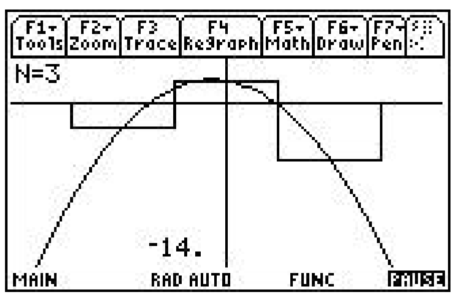
**Where necessary, estimate to 4 decimal places.**

1. (20 pts.). Without using the *allsums* program, estimate the area under the graph of $f(x) = (1 - x)(2x + 3)$ from $x = -3$ to $x = 3$ using three approximating rectangles and midpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate? Explain.

**ANS:** $\Delta x = \dfrac{b-a}{n} = \dfrac{3-(-3)}{3} = 2$; $x_0 = -3; x_1 = x_0 + \Delta x = -1; x_2 = 1; x_3 = 3$.

The midpoints are: $x_1^* = \dfrac{x_0 + x_1}{2} = -2; x_2^* = 0; x_3^* = 2$.

$M_3 = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x$
$= f(-2)\Delta x + f(0)\Delta x + f(2)\Delta x$
$= (-3)(2) + (3)(2) + (-7)(2) = -14.$




Since the function is concave down on [–3, 3], the midpoint sum is an overestimate.

2. (15 pts.). Sketch the region enclosed by $y = x^2$, $y = 8 - x^2$ and $4x - y + 12 = 0$.
Find the area.

**ANS:**

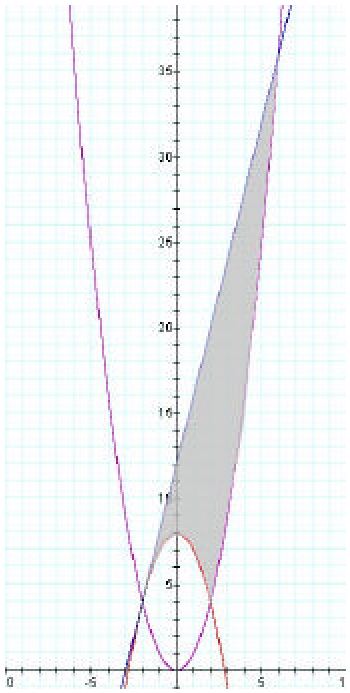


fig. 1

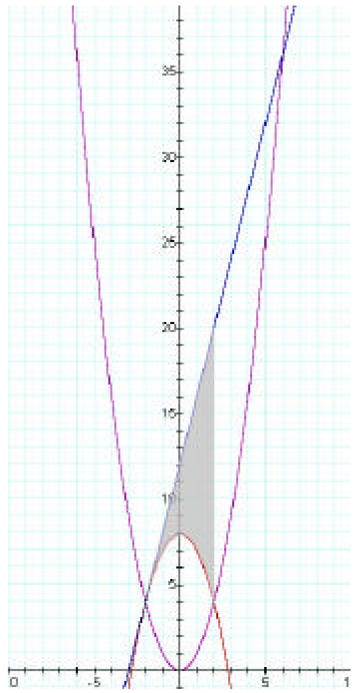


fig. 2

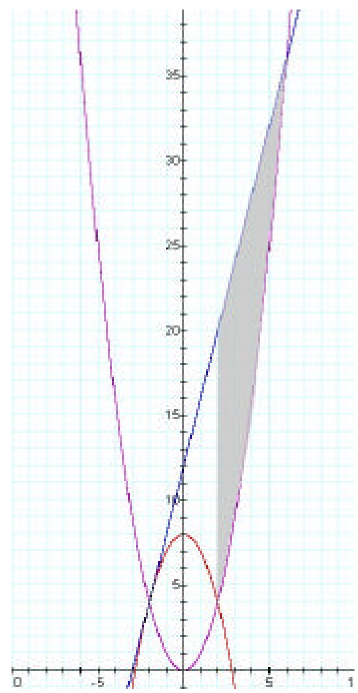


fig. 3

The area desired in shown in figure 1. There are two integrals needed: the ‘upper’ function is $4x - y + 12 = 0$; but there are two ‘lower’ functions: $y = 8 - x^2$ on [–2, 2] and $y = x^2$ on [2, 6]. The limits of integration are found using the intersection function on the calculator. So,

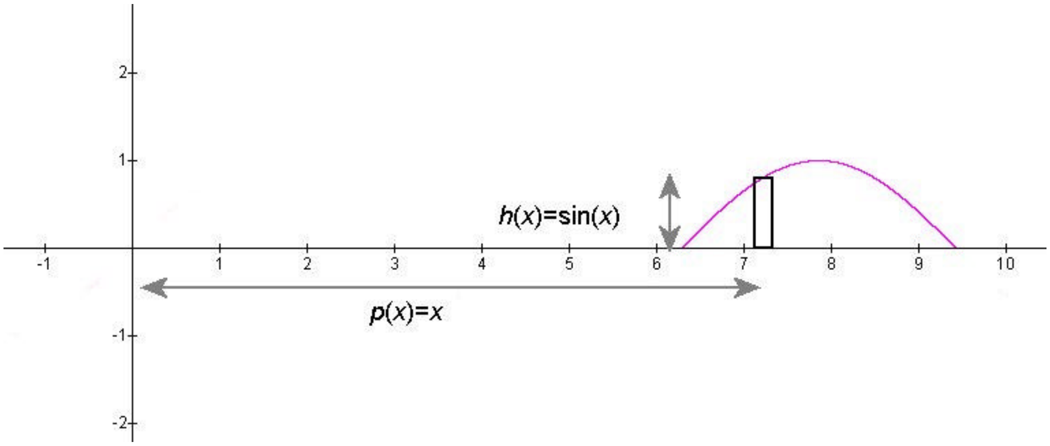
$$A = \int_{-2}^{2} \left[(4x + 12) - \left(8 - x^2\right)\right] dx + \int_{2}^{6} \left[(4x + 12) - \left(x^2\right)\right] dx$$

$$= \int_{-2}^{2} \left(x^2 + 4x + 4\right) dx + \int_{2}^{6} \left(-x^2 + 4x + 12\right) dx = \left(\frac{x^3}{3} + 2x^2 + 4x\right)_{-2}^{2} + \left(-\frac{x^3}{3} + 2x^2 + 12x\right)_{2}^{6}$$

$$= \frac{56}{3} - \left(\frac{8}{3}\right) + 72 - \frac{88}{3} = 64.$$

3. (20 pts.). Find the volume of the solid formed by revolving the region bounded by $y = \sin x$, $x = 2ð$, $x = 3ð$ and $y = 0$ about the $y$-axis. Sketch the area.

**ANS:**




Use shell method. $V = 2p\int_{2p}^{3p} x\sin x dx$. Use integration by parts:

| <u>U</u> | | <u>dv</u> |
|---|---|---|
| x | ↘ | sin $x$ |
| 1 | ↘ | –cos $x$ |
| 0 | → | –sin $x$ |

$$V = 2p\int_{2p}^{3p} x\sin x dx = 2p\left(\sin x - x\cos x\right)\Big|_{2p}^{3p}$$
$$= 2p\left[\left(\sin(3p) - (3p)\cos(3p)\right) - \left(\sin(2p) - (2p)\cos(2p)\right)\right]$$
$$= 2p\left[\left(0 - (-3p)\right) - \left(0 - (2p)\right)\right] = 2p\left[3p + 2p\right] = 10p^2.$$

4. (5 pts.). Use simpson's rule with $n = 30$ to find the length of the curve $y = \csc(2x)$, $0.5 \le x \le 1.5$.

**ANS:** $y' = -2\csc(2x)\cot(2x) = -2\dfrac{\cos(2x)}{\sin^2(2x)}$.

$$arclength = \int_{0.5}^{1.5}\sqrt{1+(y')^2}dx = \int_{0.5}^{1.5}\sqrt{1+\left(-2\frac{\cos(2x)}{\sin^2(2x)}\right)^2}dx = \int_{0.5}^{1.5}\sqrt{1+4\frac{\cos^2(2x)}{\sin^4(2x)}}dx.$$

$S_{30} = 6.6598.$

5. (10 pts.). Derive the formula for the derivative of $\sin^{-1}(x)$. Show all steps and be specific.

**ANS:** $y = \sin^{-1} x \Rightarrow x = \sin y$. The range is $\left[-\frac{p}{2}, \frac{p}{2}\right]$. Differentiate implicitly:

$1 = \cos y \cdot y'$. Solve for y′: $y' = \frac{1}{\cos y}$. Now $\cos y \geq 0$ since y $\in \left[-\frac{p}{2}, \frac{p}{2}\right]$. We also know, $\cos y = \sqrt{1 - \sin^2 y}$ which implies $\cos y = \sqrt{1 - x^2}$ because $x = \sin y$. Thus, $y' = \frac{1}{\sqrt{1 - x^2}}$.

6. (15 pts.). Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

**ANS:** $\lim_{x \to 0} \frac{\tan x - x}{x^3}$ [type $\frac{0}{0}$] $\overset{H}{=} \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$ [type $\frac{0}{0}$] $\overset{H}{=} \lim_{x \to 0} \frac{2 \sec x \cdot \sec x \tan x}{6x}$

$= \lim_{x \to 0} \frac{\sec^2 x \tan x}{3x}$ [type $\frac{0}{0}$] $\overset{H}{=} \lim_{x \to 0} \frac{2 \sec x \cdot \sec x \tan x \cdot \tan x + \sec^4 x}{3} = 1/3.$

7. (10 pts.). Find $\int_0^2 x^2 3^x \, dx$.

**ANS:** Use integration by parts:

| U | dv |
|---|---|
| $x^2$ | $3^x$ |
| $2x$ | $3^x/\ln 3$ |
| $2$ | $3^x/(\ln 3)^2$ |
| $0$ | $3^x/(\ln 3)^3$ |

So, $\int_0^2 x^2 3^x \, dx = \left[\frac{x^2 3^x}{\ln 3} - \frac{2x3^x}{(\ln 3)^2} + \frac{2 \cdot 3^x}{(\ln 3)^3}\right]_0^2$

$= \left(\frac{2^2 3^2}{\ln 3} - \frac{2(2)3^2}{(\ln 3)^2} + \frac{2 \cdot 3^2}{(\ln 3)^3}\right) - \left(\frac{0^2 3^0}{\ln 3} - \frac{2(0)3^0}{(\ln 3)^2} + \frac{2 \cdot 3^0}{(\ln 3)^3}\right) = \frac{36}{\ln 3} - \frac{36}{(\ln 3)^2} + \frac{18}{(\ln 3)^3} - \frac{2}{(\ln 3)^3}$

$= \frac{36}{\ln 3} - \frac{36}{(\ln 3)^2} + \frac{16}{(\ln 3)^3} \approx 15.008$

(5 pts.) **Answer only ONE of the following:**

8. Find $\int_e^\infty \frac{1}{x \ln x} dx$

**ANS:** $\int_e^\infty \frac{1}{x \ln x} dx = \lim_{t\to\infty} \int_e^t \frac{1}{x \ln x} dx$. Use substitution: let $u = \ln x$, $du = dx/x$. So,

$= \lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u} du = \lim_{t\to\infty} [\ln u]_1^{\ln t} = \lim_{t\to\infty} [\ln(\ln t) - \ln 1] = \infty$. Thus, the improper integral diverges.

9. Find $\int \cos^4 x \sin^3 x dx$.

**ANS:** Let $u = \cos x$; $du = -\sin x\, dx$. Now, we have
$-\int u^4 (\sin^2 x) du$. Use the trig identity, $\sin^2 x = 1 - \cos^2 x$. So, $-\int u^4 (1 - \cos^2 x) du$ and $\cos x = u$,
so $-\int u^4 (1 - u^2) du = -\int (u^4 - u^6) du = -\frac{1}{5} u^5 + \frac{1}{7} u^7 + C = -\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C$.