MAT 254 – Fall Quarter 2002
Test 4 – Answers

NAME____

Show work and write clearly.

1. (10 pts.) Derive the formula for the derivative of $\cos^{-1}(x)$.

ANS: $y = \cos^{-1}(x) \Rightarrow x = \cos(y)$. Differentiate implicitly. $1 = y'(-\sin(y)) \Rightarrow y' = -\frac{1}{\sin y}$.

Using a well-known trig identity $(1 = \cos^2 y + \sin^2 y)$, we have $y' = -\frac{1}{\sqrt{1-\cos^2 y}}$. From the first step, we know $x = \cos(y)$, so $y' = -\frac{1}{\sqrt{1-x^2}}$.

2. (30 pts.) Find the derivatives of the following:

a. $y = \cos^{-1}(\sin x)$ **ANS:** $y' = -\frac{1}{\sqrt{1-\sin^2 x}}(\cos x) = -\frac{1}{\sqrt{\cos^2 x}}(\cos x) = -1$

b. $y = \sec^{-1}(e^x)$ **ANS:** $y' = \frac{1}{e^x\sqrt{(e^x)^2 - 1}}(e^x) = \frac{1}{\sqrt{e^{2x} - 1}}$

c. $y = x^3\sqrt{1+x^2}$

ANS: You can use product and chain rules or use logarithmic differentiation. Here I'm using logarithmic differentiation.

$$\ln y = \ln\left(x^3\sqrt{1+x^2}\right) = 3\ln x + \frac{1}{2}\ln\left(1+x^2\right)$$

$$\Rightarrow \frac{1}{y}y' = 3\frac{1}{x} + \frac{1}{2}\left(\frac{1}{1+x^2}\right)(2x) = \frac{3}{x} + \frac{x}{1+x^2}$$

$$\Rightarrow y' = \left(\frac{3}{x} + \frac{x}{1+x^2}\right)y = \left(\frac{3}{x} + \frac{x}{1+x^2}\right)x^3\sqrt{1+x^2}$$

d. $y = (\ln x)^{\tan x}$ **ANS:** Use logarithmic differentiation. $\ln y = \tan x(\ln(\ln x))$. Use product rule on right-hand side. $\Rightarrow \frac{1}{y}y' = \sec^2 x(\ln(\ln x)) + \tan x\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$.

$$\Rightarrow y' = \left[\sec^2 x(\ln(\ln x)) + \tan x\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)\right](\ln x)^{\tan x}$$

e. $y = x^{(e^x)}$ **ANS:** Use logarithmic differentiation. $\ln y = e^x \ln x$. Use product rule on right-hand side. $\Rightarrow \frac{1}{y}y' = e^x \ln x + e^x\left(\frac{1}{x}\right) \Rightarrow y' = \left[e^x \ln x + e^x\left(\frac{1}{x}\right)\right]x^{e^x}$.

3. (10 pts.) One hundred fruit flies are placed in a breeding container that can support a population of at most 5000 flies. If the population grows with a constant relative growth rate of 2% per day, how long will it take for the container to reach capacity?

ANS: Use $P(t) = P(0)e^{kt}$. We know $P(0) = 100$, $P(t) = 5000$ is maximum and $k = 0.02$. So,

$$5000 = 100e^{0.02t} \Rightarrow 50 = e^{0.02t} \Rightarrow \ln 50 = 0.02t \Rightarrow t = \frac{\ln 50}{0.02} \approx 196 \text{ days.}$$

4. (40 pts.) Find the following limits:

a. $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$ **ANS:** Type 0/0 so use l'Hôpital's Rule $\Rightarrow \lim_{x\to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \frac{1}{6}$.

b. $\lim_{x\to 0^+} x\ln x$ **ANS:** Type 0·∞ so change to $\lim_{x\to 0^+} \frac{\ln x}{1/x}$. This is type ∞/∞ so use l'Hôpital's Rule $\Rightarrow \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$.

c. $\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$ **ANS:** Type 0/0 so use l'Hôpital's Rule $\Rightarrow \lim_{x\to 0} \frac{\sin x}{1+2x} = \frac{0}{1} = 0$.

d. $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$ **ANS:** Type ∞–∞ so rationalize numerator:

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)\cdot\frac{\left(x+\sqrt{x^2+x}\right)}{\left(x+\sqrt{x^2+x}\right)} = \lim_{x\to\infty}\frac{x^2-x^2+x}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}.$$

Now divide all terms by the term in the denominator with the largest exponent.

$$\Rightarrow \lim_{x\to\infty}\frac{-x/x}{x/x+\sqrt{x^2/x^2+x/x^2}} = \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+1/x}} = \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}.$$

e. $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$ **ANS:** Type 0/0 so use l'Hôpital's Rule $\Rightarrow \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}$.

This is still type 0/0, so use l'Hôpital's Rule again $\Rightarrow \lim_{x\to 0}\frac{\left(\frac{1}{2}\right)\frac{1}{2(1+x)^{3/2}}}{2} = -\frac{1}{8}$.

5. (10 pts.) Forty percent of a radioactive substance decays in 5 years. How long would it take the sample to decay to 1% of its original amount?

ANS: Use $P(t) = P(0)e^{kt}$. We use $P(0) = 100$. If 40% decays, the 60% is left after 5 years, i.e., $P(5) = 60$. So, $60 = 100e^{k(5)} \Rightarrow 0.6 = e^{k(5)} \Rightarrow \ln 0.6 = 5k \Rightarrow k = \frac{\ln 0.6}{5} \approx -0.1022$. Now to find the time it takes to decay to 1%, we use $P(0) = 100$ and $P(t) = 1$. So,

$$1 = 100e^{-0.1022t} \Rightarrow 0.01 = e^{-0.1022t} \Rightarrow \ln 0.01 = -0.1022t \Rightarrow t = \frac{\ln 0.01}{-0.1022t} \approx 22.5 \text{ years.}$$