Framework for Mathematical Proficiency for Teaching

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During the last few years, we have been attempting to design a framework for the construct of *mathematical knowledge for teaching* (MKT) as it might be applied to secondary school mathematics. Working from the bottom up, we began by developing a collection of sample situations. Each situation portrays an incident in teaching secondary mathematics in which some mathematical point is at issue. (For details of our approach, see Kilpatrick, Blume, & Allen, 2006.) Using the situations, we have attempted to identify the special knowledge of secondary school mathematics that is beneficial for teachers to have but that other users of mathematics would not necessarily need. Looking across situations, we have tried to characterize that knowledge.

Our initial characterization was much influenced by the work of Deborah Ball and her colleagues at the University of Michigan (Ball, 2003; Ball & Bass, 2000; Ball, Bass, & Hill, 2004; Ball, Bass, Sleep, & Thames, 2005; Ball & Sleep, 2007). In particular, Ball et al. have partitioned MKT into components that distinguish between subject matter knowledge and pedagogical content knowledge (Shulman, 1986). They have identified four components: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2004). And more recently, they have added two additional kinds of knowledge: knowledge of curriculum and knowledge at the mathematical horizon. An example of the latter is "being aware that two-digit multiplication anticipates the more general case of binomial multiplication later in a student's mathematical career" (Ball, 2003, p. 4). Figure 1 shows the six components and how they are related.

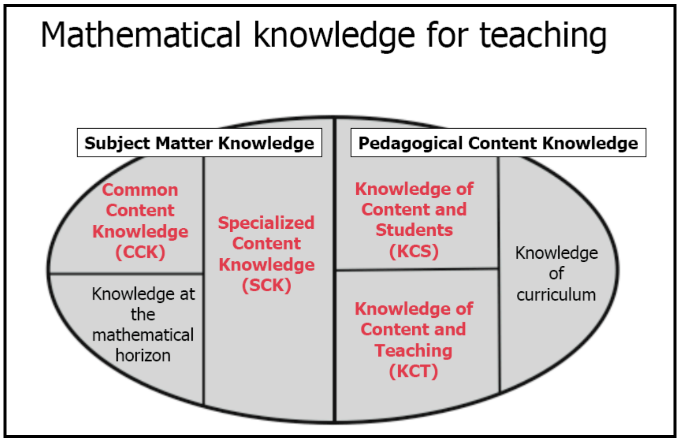

Figure 1. Model of MKT (Ball & Sleep, 2007).

As we worked on developing our own framework, we considered attempts to develop similar frameworks (e.g., Adler & Davis, 2006; Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tatto et al., 2008). We became increasingly concerned that whatever framework we developed needed to reflect a broader, more dynamic view of mathematical knowledge.

The philosopher Gilbert Ryle (1949) claimed that there are two types of knowledge: The first is expressed as "knowing that," sometimes called *propositional* or *factual* knowledge, and the second as "knowing how," sometimes called *practical* knowledge. We wanted to capture this distinction and at the same time to enlarge the MKT construct to include such mathematical aspects as reasoning, problem solving, and disposition. Consequently, we adopted the term *proficiency,* which we use in much the same way as the term is used in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001). We see the content dimension of mathematical proficiency for teaching as comprising a number of strands that go beyond a simple contrast between knowledge and understanding. We also include a teaching dimension of mathematical proficiency for teaching as a way of adding practical knowledge to factual knowledge and of capturing how teachers' mathematical proficiency is situated in their classroom practice. We have included an activities dimension of mathematical proficiency to acknowledge the teachers' proficiency in doing mathematics. It should be understood that along the three dimensions, teachers' proficiency can be at any level of development from novice to expert. Our current framework is shown in Figure 2.

1. Mathematical Proficiency with Content
    - Conceptual understanding
    - Procedural fluency
    - Strategic competence
    - Adaptive reasoning
    - Productive disposition
    - Historical and cultural knowledge

2. Mathematical Proficiency in Teaching
    - A. Preparing and Teaching a Lesson
        - Understanding and using examples and tasks
        - Understanding and translating across representations
        - Understanding and using classroom discourse
        - Knowing and using the curriculum
        - Knowing and using instructional tools and materials
    - B. Knowing Students as Mathematical Learners
    - C. Reflecting upon Practice

3. Proficiency in Mathematical Activity
    - Recognize structure and conventions
    - Connect within and outside subject
    - Represent
    - Generalize
    - Constrain
    - Model

Figure 2. Framework for mathematical proficiency for teaching.

Proficiency with Mathematical Content (PMC)

Part of Mathematical Proficiency for Teaching can be described as Proficiency with Mathematical Content (PMC). We focus on six aspects or categories of mathematical content, the knowledge of which would benefit a teacher of secondary mathematics. We describe these categories by means of stating how one might demonstrate PMC in each case, or what a person is like who has such proficiency. There is a range of proficiency in each category so that a teacher may become increasingly proficient in the course of his/her career. At the same time, certain categories may involve greater depth of mathematical knowledge than others. For example, *conceptual understanding* involves a different kind of knowledge than *procedural fluency*, though both are important. Only rote knowledge is required in order to demonstrate procedural fluency in mathematics. Conceptual understanding, however, involves (among other things) knowing *why* the procedures work.

Conceptual Understanding

This is sometimes described as the "knowing why" of mathematical knowledge. A person may demonstrate conceptual understanding by such actions as deriving needed formulas without simply retrieving them from memory, evaluating an answer for reasonableness and correctness, understanding connections in mathematics, or formulating a proof.

[Attempts have been made to further classify the levels of conceptual understanding (van Hiele).]

Some examples of conceptual understanding are:

1. knowing and understanding where the quadratic formula comes from (including being able to derive it)

2. seeing the connections between right triangle trigonometry and the graphs of trigonometric functions
3. understanding how data points can affect mean and median differently

Procedural Fluency

A person with procedural fluency knows some conditions for when and how a procedure may be applied and can apply it competently. However, procedural fluency alone would not allow one to independently derive new uses for an old procedure, such as completing the square to solve $ax^6 + bx^3 = c$. Procedural fluency can be thought of as part of the "knowing *how*" of mathematical knowledge. Such fluency is useful because the ability to quickly recall and accurately execute procedures significantly aids in the solution of mathematical problems.

The following are examples of procedural fluency:

1. recalling and using the algorithm for long division of polynomials
2. sketching the graph of a linear function
3. finding the area of a polygon using a formula
4. using key words to translate the relevant information in a word problem into an algebraic expression

Strategic Competence

Strategic competence requires procedural fluency as well as a certain level of conceptual understanding. To demonstrate strategic competence, two components are necessary: a generative function and an evaluative function. In problem solving, for example, a person must first be able to generate possible problem solving strategies (such as utilizing a known formula, deriving a new formula, solving a simpler problem, trying extreme cases, graphing, etc.), and then must accurately evaluate the relative effectiveness of those strategies. One must then accurately employ the chosen strategy to reach a solution. Strategic competence could be described as "knowing how," but it is different from procedural fluency in that it requires creativity and flexibility because problem-solving strategies cannot be reduced to mere procedures.

Specific examples of strategic competence are:

1. recognizing problems in which the quadratic formula is useful (this goes beyond simply recognizing a quadratic equation or function)
2. figuring out how to partition a variety of polygons into "helpful" pieces in order to find their areas

Adaptive Reasoning

Someone with adaptive reasoning is able to adjust to changes in assumptions and conventions. This necessitates an ability to recognize assumptions in a mathematical system, compare systems structurally, and work in a variety of systems. For example, someone who can understand the conversion between Cartesian and polar coordinates is able to compare the systems structurally. Someone who can graph both Cartesian and polar equations without necessarily comparing their representations is able to work in both systems. A person with a high level of adaptive reasoning can do both.

Adaptive reasoning includes the ability to reason both formally and informally. Some examples of formal reasoning are using rules of logic (necessary and sufficient conditions, syllogisms, etc.) and structures of proof (by contradiction, induction, etc.). Informal reasoning may include creating and understanding appropriate analogies, utilizing semi-rigorous justification, and reasoning from representations.

Examples of adaptive reasoning are:

1. operating in more than one coordinate system
2. proving an if-then statement by proving its contrapositive
3. determining the validity of a proposed analogy

Productive Disposition

Those with a productive disposition believe they can benefit from engaging in mathematical activity, and are confident that they can succeed in mathematical endeavors. They are curious and enthusiastic about mathematics, and are therefore motivated to see a problem through to its conclusion, even if this involves thinking about the problem for a long time in order to make progress on it. Those with a productive disposition are able to notice mathematics in the world around them, and apply mathematical principles to situations outside the mathematics classroom. (Cuoco, 1996)

Examples of productive disposition are:

1. noticing symmetry in the natural world
2. persevering through multiple attempts to solve a problem

Historical and Cultural Knowledge

Someone with knowledge of the history of mathematics often has a better understanding of the origin and significance of various mathematical conventions. This, in turn, may increase his/her conceptual understanding of mathematical ideas, such as those linked to notation. For example, knowing that the integral symbol is an elongated *s*, from the Latin *summa* (meaning *sum* or *total*) may provide insight about what the integral function is.

Cross-cultural knowledge (i.e. awareness of how people in various cultures or even in various disciplines conceptualize and express mathematical ideas) may have a direct impact on mathematical understanding. For example, one may be used to defining a rectangle in terms of its sides and angles. However, people in some non-Western cultures define a rectangle in terms of its diagonals. Being able to conceptualize both definitions can strengthen one's mathematical proficiency.

The following are more examples of historical and cultural knowledge:

1. being familiar with the historic progression from Euclidean geometry to multiple geometric systems
2. being able to compare mathematicians' convention of measuring angles counterclockwise from horizontal with the convention (used by pilots, ship captains, etc.) of indicating directions in terms of degrees away from North
3. understanding similarities and differences in algorithms typically taught in North America and those taught elsewhere

Proficiency in Mathematics Teaching (PMT)

Proficiency in mathematics teaching requires that teachers have a working knowledge of key topics that inform the practice of teaching. Using Ryle's (1949) terminology, proficiency in mathematics teaching can be thought of more as the "knowing how", whereas proficiency in mathematics content is more of the "knowing that". When discussing proficiency in mathematics teaching, three broad categories emerge – knowing how to prepare and teach lessons, how to interact with students, and how to conduct assessment. The first category addresses the type of knowledge teachers need in order to effectively guide and encourage productive lessons. In essence, teachers of secondary mathematics should not only have a strong content knowledge base, but their conceptual understanding of the material should be formed in such a way that they can facilitate their students' own development of mathematical understanding. The second category is the teacher's understanding the students. When students enter the secondary classroom, they bring with them a vast amount of previous mathematical experiences. To possess proficiency in mathematics teaching, teachers should be aware of those experiences and how they have influenced their students mathematically, both in terms of ability and emotion, and create opportunities for their students to grow in content knowledge. The last category would be the reflection in which teachers think about previous experiences. Possessing proficiency in mathematics teaching would allow for teachers to use their content knowledge to determine the effectiveness of a lesson both in terms of the ability to grow in conceptual understanding of the students and to address the recommendations established through research and regulation. Although there are many items that seem quite general, we are going to investigate these ideas from the mathematical perspective.

Preparing and Teaching a Lesson

Understanding and using examples and tasks. Selecting or constructing instructionally powerful examples is a common and important activity of teaching mathematics. One must come up with an example, non-example, or counterexample to address the concept at hand, without introducing unnecessary ambiguity. A teacher must also recognize the value or limitations of examples that others may introduce. Mathematical tasks are those that serve as opportunities for students to learn mathematical concepts. The teacher is concerned with selecting or creating tasks that are appropriate to the level of mathematics of the class, and also keeping high standards for the level of cognitive demand for students.

Understanding and translating across representations. Teachers use representations throughout the course of the day to help students learn. Students represent their work and their answers with words and pictures. Understanding and translating across representations has been a key element in many (some?) of our situations and foci. When students present solutions to tasks that are both correct and yet different, a teacher will be able to connect those two representations. When deciding upon representing a function in a lesson plan, a teacher will be able to choose the representation (or representations) which will have the best chance to illustrate the function. Connecting between a physical model, such as a line drawn on a board or a plastic Platonic solid, and their mathematical counterparts requires some finesse. We have found evidence in our situations that some representations generated by students, while seemingly strong models of the concept, present difficulties and possible barriers to the mathematical understanding of the concept.

Understanding and using classroom discourse . The intercommunication between and among students and teachers is vital. Classroom interactions play a significant role in teachers' understandings of what their students know and are learning. Examining classroom discourse can reveal how both students and teachers understand and make connections between the mathematical ideas being discussed. It is through discourse that implicit mathematical ideas are exposed and can be made more explicit. In order for this to happen effectively, teachers can benefit from an understanding of discourse on both the theoretical and practical levels. Theoretical understanding guides the teachers' understanding of the importance of appropriate discourse practices. Reading and incorporating what is learned from research on discourse provides the teacher with additional information about incorporating discourse into practice. Building a practical understanding of, and knowledge base of actions for, engaging students in discourse about important mathematical ideas informs and guides teaching practice and enhances the impact and usefulness of the practice for teachers and learners alike. One simple example would be specificity in the use of appropriate mathematical language in the classroom.

Knowing and using the curriculum. How mathematical knowledge is used to teach mathematics in a specific classroom, or with a specific learner, or a specific group of learners is influenced by the curriculum that organizes the teaching and learning. A teacher's mathematical proficiency can make the curriculum meaningful, connected, relevant, and useful. For example, a teacher who is mathematically proficient can think of teaching the concept of area as part of a curriculum that includes ideas about measure, descriptions of two-dimensional space, measures of space under a curve, measures of the surface of three-dimensional solids, infinite sums of discrete regions, operations on space and measures of space, and useful applications involving area. This is a very different perspective of the curriculum from someone who thinks of area in terms of formulas for polygonal regions.

Mathematical proficiency for knowing and using the curriculum in teaching requires a teacher to identify foundational or prerequisite concepts that enhance the learning of a concept as well as how the concept being taught can serve as a foundational or requisite concept for future learning. A teacher needs to know how a particular concept fits within a student's learning trajectory. At the same time, proficient mathematics teachers understand that there is not a prescribed, linear order for learning mathematics, but rather multiple mathematical ways to approach a concept and to revisit a concept. Mathematical concepts and processes evolve in the learner's mind becoming more complex and sophisticated with each iteration. Mathematical proficiency prepares a teacher to build a curriculum that not only connects mathematical ideas but also builds a disposition within students where they expect mathematical ideas to be connected (Cuoco, 2001).

A mathematically proficient teacher understands that a curriculum contains not only mathematical entities but also mathematical processes for relating, connecting, and operating on those entities (NCTM Standards, 1989, 2000). A teachers must have mathematical proficiency to set appropriate curricular goals for their students (Adler, 2006). For example, a teacher needs mathematical knowledge to select and teach functions that help students build a basic repertoire of functions (Even, 1990).

Knowing and using instructional tools and materials. When determining the set of instructional tools that teachers might implement in their classroom, digital technology, like graphing calculators and computer software comprises part of that list. The use of this technology allows students to see many examples and representations of a concept in a short period of time. Although teachers need not have a background in programming the technology, there is mathematics involved in the implementation of technology in the classroom.

Instructional tools are not limited to electronic technology. MPT can be found embedded in the choice of manipulatives that teachers use as part of their lessons. In selecting manipulatives and other visual aids, the teacher would need to address the mathematics in choosing certain manipulatives and the extensions that can be made when they are implemented as part of a daily lesson. For example, the choice of algebra tiles allows the students to visualize the factoring of a quadratic trinomial as finding the area of a rectangle. Although this method provides a visualization for quadratic trinomials that are the product of two linear binomials with rational coefficients, the difficulty of the visualization increases if the quadratic trinomial does not yield the two linear factors with rational number coefficients.

Knowing Students as Mathematical Learners

Every student brings his or her own previous mathematical experiences into a mathematics classroom and each lesson provides the student an opportunity to grow their mathematical knowledge through a variety of learning tasks.

Each lesson and task provides the teacher with opportunities to observe and correct common student errors, determine topics that students do understand, and identify concepts needed for understanding more advanced mathematical concepts. An example of a student error in an Algebra I class is the claim that the trinomial $x^2 + 5x + 6$ is the product of the binomials $(x + 5)(x + 1)$. Likewise, students could have an understanding of a topic and then over-generalize. For example, a student could demonstrate that the multiplication of real number is commutative. The teacher in a high school algebra class would need to address that the multiplication of matrices with real number entries, however, is not commutative. In addition, there are instances in which a procedural understanding inhibits the development of an understanding, be it procedural or conceptual, of a second concept. In our Situations work, we see that treating exponentiation as repeated multiplication creates conflict for students when explaining the expression of a real number base and an irrational number exponent.

Reflecting Upon Practice

There are a wide variety of ways to reflect upon one's teaching and bring to bear one's MPT upon that reflection. Many of these forms of reflection focus on the application and use of mathematics in the work. Some of them, however, may involve more objective approaches, such as examining video records or working with another teacher, teacher-educator, or administrator to look at an individual's teaching. It is vital for teachers to review the work they have done and/or the results of that work, such as evidence of students' mathematical learning. They can then evaluate it for effectiveness. Evaluation may be done at any point in the process of teaching, not just after a lesson has been completed (or taught). For instance, as a teacher is engaged in teaching a lesson, he should monitor his activity, including the reactions of the students, to see if the work is effective, *i.e.* there is evidence of student mathematical learning, and, if not, to consider an alternative course of action that might result in increased student understanding of the mathematics. The teacher could also use the foci of a published situation similar to the work done in the classroom as a guide for reflecting on their own work. Reflection, whatever form it takes, is one of many tools the teacher uses to be more aware of his or her practice. The information gathered during this process should inform an understanding of a teacher's mathematical practice and allow him or her to continuously improve it.

Proficiency in Mathematical Activity (PMA)

Googling the phrase "Mathematics is not a spectator sport" returns approximately 3100 hits. The phrase implies that mathematics is something one does, and not something one watches. Proficiency in Mathematical Activity acknowledges that mathematical knowledge has a dynamic aspect by describing actions taken upon mathematical objects. Mathematical objects include functions, numbers, matrices and so on. One might think of them as the nouns of mathematics. The categories listed below describe the actions one uses with these mathematical objects.

Recognize Structure and Conventions

Until the eighth grade, school mathematics usually deals with integers and rational numbers. Although these sets have their own algebraic structure, at the secondary level, more mathematical structure is introduced at a faster rate. The structure of algebra reveals itself as students move from the study of rational numbers to the study of real and complex numbers, variables, and functions. Operations once performed only on rational numbers are extended to these new objects such as polynomials. New operations such as the inverse function and composition of functions are introduced. In geometry, analytic and other non-Euclidean geometries are introduced. With these structures come new conventions. In algebra, new notation such as $(f \circ g)(x)$ and summation notation succinctly portray actions done. Familiar notation such as x^{-1} and f^{-1} is used in different ways depending upon context. Familiar operations sometimes do not have the same meaning with new structures, e.g. matrix multiplication. Definitions are conventions as well. If one chooses to restrict trapezoids to only one pair of parallel sides, then the structure of geometry following a definition is affected by this convention.

Connect Within and Outside the Subject

Secondary mathematics introduces more content from more branches of mathematics. Systematic study of geometry, algebra, statistics, probability and the calculus extends students' mathematical knowledge. *Connecting within mathematics* means that teachers should have a working knowledge of both the development of a mathematical topic and how that topic relates to other areas of mathematics. For example, students may study transformations using paper folding to investigate reflections, rotations, translations, and glide reflections. When studying the Cartesian plane, a teacher should be able to quantify the transformations. Matrices and matrix operations can be used to transform one figure into another. Connecting within mathematics also means to be able to connect student-generated algorithms to the standard algorithm. Rewriting equations from the Cartesian coordinate system into polar coordinates goes beyond being able to work in both systems and illustrating the similarities between the two. Another example is connecting variables and numbers. How does the expression $y = 3x^2 - 8x + 2$ relate to the numbers that might be substituted for x? What sort of numbers does that expression generate if the domain of x changes? This connects to functions and the study of those objects. How does factoring a quadratic connect to factoring a number?

Connecting to areas outside of mathematics requires teachers to look for mathematics outside of their classroom, both within and beyond the boundaries of the school walls. Continuing with transformations, the video games Doom and Quake used a "graphics engine" which uses matrix operations to generate the images on the screen. The Federalist Papers were written by three different people using the pseudonym Publius. Statistical tests have been used to estimate who authored which paper. Connecting within and without mathematics means both looking for applications for mathematics as well as situations from which to extract mathematics.

Represent

At all levels of mathematics, teachers represent mathematical objects using different methods. Mathematical objects are described using numbers, symbols, pictures, words, physical objects, and other means. Mathematical objects can be represented by many means, and each representation affords a person different views of the mathematical object. A verbal representation might be a mnemonic such as SOHCAHTOA or a short description such as "slope may be thought of as rise over run." Tables, symbols or graphs can represent functions. Although a table might be the best representation when trying to find values of a function, a graph could provide evidence a function is 1-1. In a geometry class, a teacher might have physical objects for quadrilaterals and other shapes while also having students explore what conditions must be met to create similar shapes on a program such as Geometer's Sketchpad. Teachers use analogies and language to describe functions as well, using function machines or other analogies to impart some of the qualities of a function. Teachers who can represent well should be able to switch smoothly between representations and know that each representation emphasizes different aspects of the same object.

Generalize

Cuoco (1996) argues "mathematicians talk small and think big." Teachers who generalize are able to test conjectures, expand the domains of rules and procedures, and adapt mathematical ideas to new situations. If a conjecture is made in a classroom, a teacher will be able to test the conjecture with different domains or sets of objects. For example, a student may state multiplication returns a number equal to or larger than either initial factor. A teacher should be able to test if such a conjecture holds true in every domain. Similarly, a teacher should be able to explain why rules may or may not work under new domains. Although one might demonstrate the exponent rules with rational exponents, one should also be able to demonstrate why such rules still work for complex exponents. Processes may also be generalized.

Constrain

To constrain in mathematics means to define the limits of a particular mathematical idea. When finding the inverses of functions, one must sometimes constrain the domain if one wants the inverse to be a function as well. The inverse of $f(x) = \sin x$ is only a function if the new domain is restricted. Constraints can be removed or replaced to explore new mathematics. When mathematicians tinkered with the constraint of Euclid's fifth postulate, new geometries were formed. When one removes the constraint of the plane when using Euclidean figures, the mathematics being used may also change as well. Another example of using constraints is when one constrains the domain within one works. Some geometric proofs are simple within coordinate geometry. However, if one constrains proving to synthetic geometry, then other techniques must be used that might display different mathematics than would be seen if the proof was only performed one way.

Model

Schoenfeld (1994) describes a modeling process that starts with a real world problem and translates the problems into a formal mathematical system. Within the formal system, one manipulates the model until a solution is found. The solution is mapped back to the real world to be tested within the real world problem. Schoenfeld notes that if "either of these mappings [to and from the formal system] are flawed, then the analysis is not valid" (p. 700). Writing an equation for a word problem is one type of modeling. For example, consider this word problem: *A class of 45 students rents vans for a field trip. Let x be the smallest number of vans needed for the trip. If each van holds 14 students, write an equation for x?* One model for this is $x = 45/14$ or 3 ¼ vans. One interpretation of this model is the class trip needs 4 vans, since 3 ¼ does not make sense when dealing with vans. However, other interpretations of this solution exist. If cars *and* vans are available, the answer could be some combination of the two. Modeling can also be seen as a recursive process. If a solution does not work within the real word context, aspects of the model, such as initial conditions are assumptions, could be changed to form a new model. Programs such as Geometer's Sketchpad allow geometrical models to be created to test hypotheses. Statistical modeling provides predictions when dealing with data points. Monte Carlo simulations model outcomes using random inputs. We also make the distinction that modeling involves a context outside of mathematics as compared to representing which resides wholly within mathematics.

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