MAC-CPTM Situations Project

Situation 43: Can You Circumscribe a Circle about This Polygon?

Prompt

In a geometry class, after a discussion about circumscribing circles about triangles, a student asked, “Can you circumscribe a circle about any polygon?”

Commentary

A polygon that can be inscribed in a circle is called a cyclic polygon. Not every polygon is cyclic, but there are infinite cyclic polygons. This can be understood by considering a given circle and all the possibilities of how many points can be placed on the circle, and then connected to form a polygon. However, there are certain classes of polygons that are noteworthy because they are always cyclic. The conditions under which a circle circumscribes a given polygon are dependent upon the relationships among the angles, the sides, and the perpendicular bisectors of the sides of the polygon. The following foci describe classes of cyclic polygons in order of the number of their sides: triangles, certain quadrilaterals, and regular polygons. Focus 3 provides one way to check whether a given polygon is cyclic: a polygon is cyclic if and only if the perpendicular bisectors of all of its sides are concurrent. Though the inclusion of various geometries would provide interesting discussion, the Foci in this Situation are limited to Euclidean geometry in a plane.

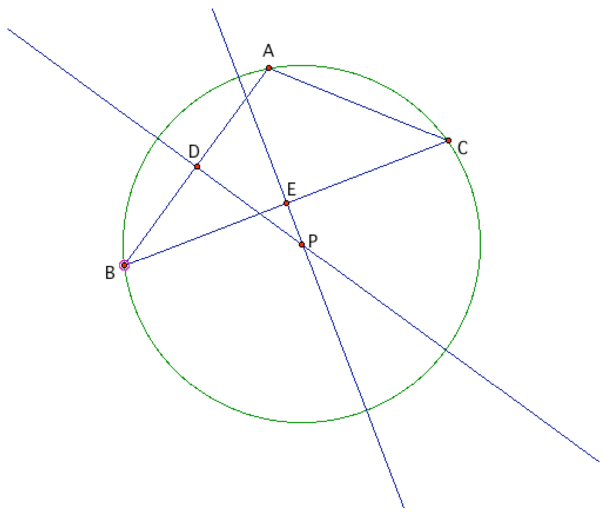
Mathematical Foci

Mathematical Focus 1

Every triangle is cyclic. This fact is core to establishing a condition for other polygons to be cyclic.

Because the center of a circle is equidistant from all the points on the circle (this distance is the radius), an inscribed triangle is one in which the three vertices of the triangle lie on the circumscribed circle. Conversely, the circle circumscribed around a particular triangle must have as its center the point that is equidistant from the vertices of the triangle. Circumscribing a circle about a triangle, then, requires finding a point that is equidistant from the three vertices of the triangle. This point is called the circumcenter of the triangle.

A point is equidistant from two points, A and B, if and only if it lies on the perpendicular bisector of the segment whose endpoints are A and B. A proof of this theorem is included in the Post-Commentary. Because of this, consider perpendicular bisectors of the sides of a triangle to find the circumcenter. Given ΔABC, the perpendicular bisectors (in the plane of ΔABC) of segments AB and BC intersect the sides at D and E, respectively. $\overline{AB}$ and $\overline{BC}$ are not parallel, so lines that are perpendicular to them are not parallel. Therefore the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ must intersect at some point, call it P. P is equidistant from A and B because it lies on the perpendicular bisector of $\overline{AB}$, and P is equidistant from B and C because it lies on the perpendicular bisector of $\overline{BC}$. So P is equidistant from A, B, and C. That is, P is the circumcenter of ΔABC. So, given any set of three noncollinear points, we can find a circumcenter.



Another way to think about triangles being cyclic is to consider the equation of a circle in the coordinate plane: $(x-h)^2+(y-k)^2=r^2$ where (h,k) is the center of the circle, r is the radius, and every ordered pair (x,y) that satisfies the equation lies on the circle. In order to find a particular circle (that is, in order to find the three unknowns h, k, and r), one would need three equations. That is, if one had three ordered pairs (x,y) (i.e. three points), one could determine the circle. This is another way of showing that three noncollinear points determine a unique circle. Given those three points, one could find the circumcenter.

Mathematical Focus 2

A convex quadrilateral in a plane is cyclic if and only if its opposite angles are supplementary.

We can establish that a convex quadrilateral is cyclic if and only if its opposite angles are supplementary. A convex quadrilateral is described as a quadrilateral in a plane such that no two points inside the quadrilateral can be connected by a segment that intersects one of the sides. Proving that two conditions (a quadrilateral being cyclic and its opposite angles being supplementary) are equivalent requires proving an implication and its converse. That is, to prove

$A \Leftrightarrow B$ we need to prove $A \Rightarrow B$ and $B \Rightarrow A$. In this proof, we use a logically equivalent construction of proving $A \Rightarrow B$ and then proving $\text{not } A \Rightarrow \text{ not } B$.

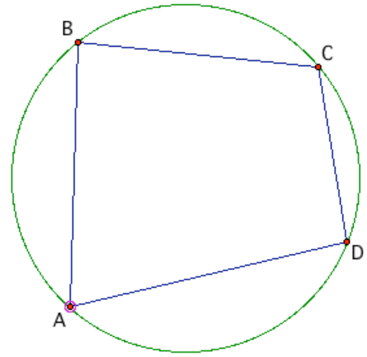
a) First we prove that given a convex, cyclic quadrilateral, its opposite angles are supplementary. In the cyclic quadrilateral ABCD shown in the following diagram, ∠ABC is opposite ∠CDA. Since the measure of an inscribed angle is half of the measure of the arc in which it is inscribed,

m∠ABC = (1/2)(m arcCDA) and m∠CDA = (1/2)(m arcABC)

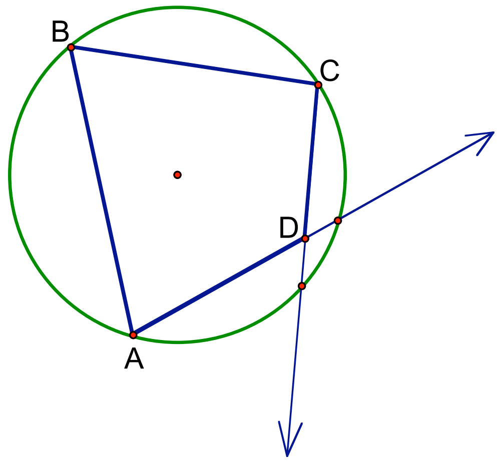
m∠ABC + m∠CDA = (1/2)(m arcCDA) + (1/2)(m arcABC)

= (1/2)(m arcCDA + m arcABC)

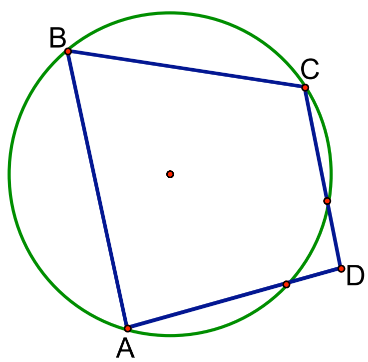
Since arcs CDA and ABC together form a circle, m arcCDA + m arcABC=360. By substitution, m∠ABC +m ∠CDA = (1/2)(360) = 180. Therefore ∠ABC and ∠CDA are supplementary. (Angles BAD and DCB could be handled similarly.)



b) Next we prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic. Begin with convex quadrilateral ABCD such that angles BAD and DCB are supplementary and angles ABC and CDA are supplementary. Draw the circle defined by points A, B, and C. (This circle can be constructed since three points determine a circle—see Focus 1). Suppose D is located in the interior of the circle. Then extend segments AD and CD until they each intersect the circle. By the inscribed angle theorem, the sum of angles BAD and DCB is less than 180 degrees (because together they subtend less than a whole circle). This is a contradiction, therefore D cannot be inside the circle.



Suppose instead that D is outside the circle. In this case, the sum of angles BAD and DCB will be greater than 180 because together they subtend more than a whole circle (they subtend the whole thing plus a piece of it twice). This is a contradiction, so D cannot be outside the circle.



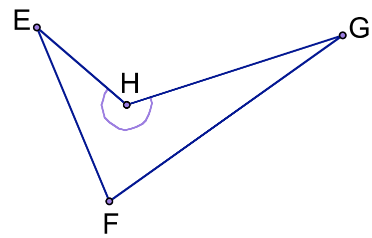
Therefore D must be on the circle and quadrilateral ABCD is cyclic.

A corollary that is implied by the above result is that every rectangle is cyclic. Also, no parallelograms other than rectangles are cyclic. Moreover, every isosceles trapezoid is cyclic. [Note: A commonly used definition of trapezoid is that it is a quadrilateral with exactly one pair of parallel sides. However, trapezoids are defined by some sources as a quadrilateral with at least one set of parallel sides. If trapezoids are defined this way, then every rectangle is an isosceles trapezoid.]

Mathematical Focus 3

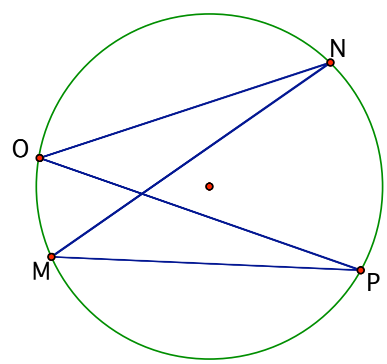
There are four-sided figures in the plane that behave differently from convex quadrilaterals. Concave quadrilaterals are never cyclic, and a four-sided figure with non-sequential vertices is cyclic if and only if its “opposite” angles are congruent.

Cyclic quadrilaterals have opposite angles that are supplementary (see Focus 2). Consider a concave polygon like quadrilateral EFGH shown below. To see that quadrilateral EFGH is non-cyclic, suppose that we construct the circle passing through the points EFG (in general, through the 3 points that occur at the polygon's interior angles that are not greater than 180 degrees – this is possible because a concave quadrilateral will have exactly one interior angle greater than 180 degrees, in this case it's angle EHG). Suppose point H lies on this circle. But then angle EHG is less than 180 degrees because it cannot subtend the whole circle (much less even more than it) and so we have a contradiction.



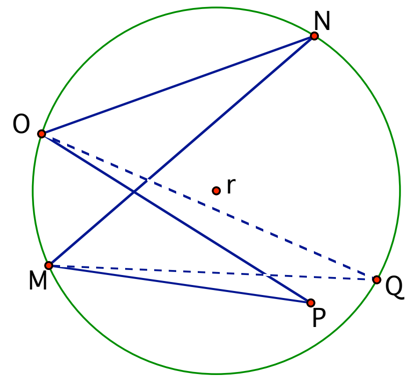
A quadrilateral is commonly defined as a polygon with four sides. Therefore it is important that the definition of polygon be clear. If the definition of polygon requires that it be a simple, closed figure (as it is in many high school mathematics textbooks), then the figures in the following discussion are not polygons, and therefore not quadrilaterals. However, if the definition requires only that a quadrilateral be a closed figure in a plane with four straight sides, then a quadrilateral with non-sequential vertices is worth discussing in this Situation.

A quadrilateral with non-sequential vertices is cyclic if and only if its "opposite" angles are congruent. Such a quadrilateral has sides that cross, as seen below. By "opposite" angles we mean, for example, angles NMP and PON or angles ONM and MPO.



In this case, we must first prove that if a quadrilateral with non-sequential vertices is cyclic, then its "opposite" angles are congruent. Consider quadrilateral MNOP above. Since angles NMP and PON lie on the circle and intersect the same arc (arc NP), they are congruent. In the same way, angles ONM and MPO both intersect arc MO, so they are congruent. It is interesting to note that the sum of the interior angles of this type of quadrilateral is not 360.

Next, we must prove that if "opposite" angles of a quadrilateral with non-sequential vertices are congruent, then the quadrilateral is cyclic. The same strategy that was used for the converse in part a) can be used here: Begin with quadrilateral MNOP such that angles NMP and PON are congruent, angles ONM and MPO are congruent, and all the vertices except P lie on circle r. If P is inside circle r, then extend sides MP and OP to intersect circle r. Then angles NMP and PON subtend different size arcs and so are not equal, which is a contradiction. If P is outside circle r, then side MP intersects circle r in a different point than side OP and so angles NMP and PON are again non-congruent, which is a contradiction. Therefore, P must lie on circle r.



Mathematical Focus 4

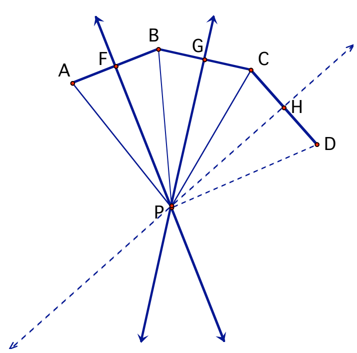
Every planar regular polygon is cyclic. However, not every cyclic polygon is regular.

As was discussed in Focus 1, a point is equidistant from two points, A and B, if and only if it lies on the perpendicular bisector of the segment whose endpoints are A and B. In order for a polygon to be cyclic, there must be a single circle that passes through all of its vertices. In other words, there must be a single point that is equidistant from all the vertices. This point must lie on the perpendicular bisectors of all the sides of the polygon. In order for a point to lie on all these bisectors, the bisectors must be concurrent, and the point of concurrency will be the circumcenter of the polygon. Since the statement about equidistance and the perpendicular bisector is biconditional, we can also make a biconditional statement about the concurrency of the perpendicular bisectors of the sides of a polygon. That is, the perpendicular bisectors of the sides of a polygon are concurrent if and only if the polygon is cyclic.

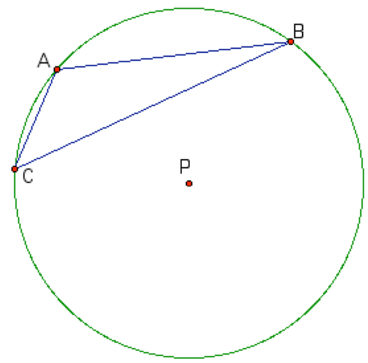
Every triangle is a cyclic polygon, as was seen in Focus 2. The question remains as to which other polygons are cyclic. By examining the perpendicular bisectors of the sides of a polygon, one can determine a set of conditions on a polygon that is sufficient to conclude whether a circle can circumscribe that polygon. In particular, we can show that every regular polygon is cyclic.

We have established that if the perpendicular bisectors of the sides of a polygon are concurrent, the polygon is cyclic. We will now show that the perpendicular bisectors of the sides of a regular polygon are concurrent and conclude that every regular polygon is cyclic.

By definition, a regular polygon is an equilateral and equiangular *n*-sided polygon. Consider a regular polygon with adjacent vertices, A, B, C, and D. Let P be the point of intersection of the perpendicular bisectors ($\overline{FP}$ and $\overline{GP}$, respectively) of $\overline{AB}$ and $\overline{BC}$. It can be shown that ΔAFP ≅ ΔBFP ≅ ΔBGP ≅ ΔCGP using the fact that P is equidistant from A, B, and C, and using the HL (hypotenuse-leg) congruence theorem. Construct $\overline{PH}$ perpendicular to $\overline{DC}$ and consider ΔHCP. ∠FBG ≅ ∠GCH since the polygon is equiangular. ∠FBP ≅ ∠GCP because ΔBFP ≅ ΔCGP. By angle subtraction, ∠GBP ≅∠HCP, so ΔFBP ≅ ΔHCP by AAS. Because of congruent triangles, HC = FB and because the polygon is equilateral, AB = CD. Also, FB=(1/2)AB since F is the midpoint of AB. So HC = (1/2)CD by substitution. We know that C, H, and D are collinear, so HC = HD. Thus $\overline{PH}$ is the perpendicular bisector of $\overline{DC}$. The argument can be extended to successive vertices of the polygon, resulting in establishing that each of the perpendicular bisectors of the sides contains the point P. That is, the perpendicular bisectors are concurrent. Note that this argument does break down if the polygon in question is not regular. If the polygon is not regular (specifically if the sides are not equal to each other) we cannot prove that all of the triangles listed above are congruent.



Therefore, every regular polygon has concurrent perpendicular bisectors, and therefore is cyclic. While every regular polygon is cyclic, it is not true that every cyclic polygon is regular. We have already seen that every triangle is cyclic, but every triangle is not an equilateral triangle and thus not regular. So it is possible to have polygons that are not regular, but are cyclic as shown in the figure below.

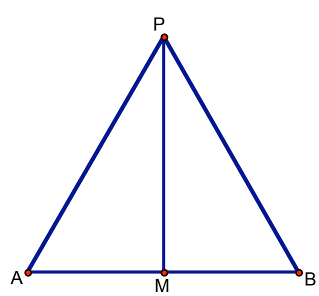


Post-Commentary

In making mathematical statements, it is important to recognize which are biconditional and which are not. In Focus 2, we proved a statement and its converse in order to prove the biconditional. In Focus 3, the converse of the statement we proved is not true and so the biconditional is not true. In Focus 1, the use of the biconditional was not as overt. In that Focus, we used the property that each point on a perpendicular bisector of a segment is equidistant from the endpoints of the segment. Later in the focus, in order to establish uniqueness, we needed the result that points that are equidistant from the endpoints of a segment lie on the perpendicular bisector of the segment. Since this statement was the converse of the earlier property, it requires a proof. We include that proof here:

A line is a perpendicular bisector of a segment AB if and only if it is the set of all points that are equidistant from A and B.

a) If a point lies on the perpendicular bisector of a line segment AB, then it is equidistant from A and B.
Let M be the midpoint of segment AB and let P be a point that lies on the perpendicular bisector of AB such that P does not lie on AB. By the definition of perpendicular bisector, AM = BM and $\angle BMP = \angle AMP = 90$. Now consider ΔAMP and ΔBMP. These triangles share side PM, so they are congruent by SAS. Therefore PA = PB.



b) If a point is equidistant from A and B then it lies on the perpendicular bisector of the line segment AB.

Let P be a point not on segment AB such that PA = PB. This means ΔPAB is isosceles and its base angles are congruent, so ∠BAP = ∠PBA. Let M be the midpoint of segment AB. Since triangles PAM and PBM share side PM, they are congruent by SAS. Since ∠BMA is a straight angle, ∠PMA + ∠BMP = 180. Also, ∠PMA = ∠BMP since ΔPAM ≅ ΔPBM. Thus,

∠PMA + ∠BMP = 180

2∠PMA = 180

∠PMA = 90

Therefore segment PM is a perpendicular bisector of segment AB.