MAC-CPTM Situations Project

Situation 21: Exponential Rules

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Prompt

In an Algebra II class, students had just finished reviewing the rules for exponents. The teacher wrote $x^m \bullet x^n = x^5$ on the board: and asked the students to make a list of values for m and n that made the statement true. After a few minutes, one student asked, "Can we write them all down? I keep thinking of more."

Commentary

The student's question goes beyond discussing the basic rules for exponents into issues of domain in the various number systems. Certainly, the key concept here, the exponent rule $x^m \cdot x^n = x^{m+n}$, is applicable and is discussed in Focus 1. As students progress through school mathematics, they extend this rule beyond the original context, positive bases and exponents, into using broader number systems, such as the set of integers or rational numbers. As one moves into other number systems, domain issues arise that must be addressed before applying any rules. For example, the values of m and n are limited, and their domains depend on the value of x. Conversely, depending on the values of m and n, the domain of x is limited. These last two statements are the themes of Focus 2 and Focus 3, respectively. The ideas in the foci will be presented algebraically, graphically, and numerically.

Mathematical Foci

Mathematical Focus 1
Mathematically precise language clarifies the rule as relating to expressions with positive bases and real exponents.

The exponent rule $x^m \cdot x^n = x^{m+n}$ may be applied to the equation $x^m \bullet x^n = x^5$ if the value of x is restricted to positive numbers. That is, if $x > 0$, and m and n are real

numbers, then $x^m \cdot x^n = x^{m+n}$. This is the commonly used statement in school mathematics.

In this case there are various things to consider in addressing the original task of determining possible values for m and n. If their values are restricted to natural (counting) numbers, then there is a finite number of solutions for (m,n): (1,4), (2,3), (3,2), and (4,1). If m and n are not limited to the natural numbers, then there are infinite solutions for (m,n), since there are infinite solutions to the equation $m + n = 5$. This is not to say that m and n could be any numbers, only that they may be any two values whose sum is 5.

The restriction $x > 0$ is an important one if we are limiting ourselves to the set of real numbers. If, for example, $m = 1.5$; then, $x^m = x^{1.5} = x^{3/2} = \sqrt{x^3}$. For this result to be a real number, x cannot be negative. If the restriction to real numbers were not in effect, x could be any number.

Mathematical Focus 2
The set of possible values for the exponent, m, is determined by the domain of the base, x.

For x^m, the value of x determines the domain for m. (All statements about x^m also hold for x^n.) Specifically, if x and m are real numbers, then:

   a. If $x = 0$, then $m > 0$. Note that 0 raised to a negative power (ex. 0^{-2}) is undefined. 0^0 is unique as well. In most cases, it is considered to be undefined. However, there are instances (for example, in the Binomial Theorem) in which 0^0 is defined as 1.
   b. If $x > 0$, then m could be any real number. This is what is assumed in the rule $x^m \cdot x^n = x^{m+n}$ and was discussed in Focus 1.
   c. If $x < 0$, then m must be a rational number (in lowest terms) whose denominator is odd.

Here are some examples that may clarify this last point. First, consider a subset of the rational numbers with odd denominators: the set of integers.

Let $x = -2$. With this (or any negative real number) as a base, the exponent, m, may be any integer and the result will be a real number:

| m	$(-2)^m$
-3	-0.125
-2	0.25
-1	-0.5
0	1

1	-2
2	4
3	-8

The exponent, *m*, may also be any rational number (in lowest terms) as long as its denominator is odd.

Consider the case when *m* is a rational number with an even denominator, such as ½; recall that the exponent now indicates an even root. An even root of a negative number yields an imaginary result. If we restrict the domains of *x* and *m* to the real numbers, an even root of a negative number lies outside the set of real numbers. If we were not limited to the real domain, *m* could be any number as long as $x \neq 0$ (as explained in a. above).

Examining a list of values of $(-2)^m$ and the resulting graph also provides insight into these restrictions. The following graph is the result of plotting the points in the table below. These points were chosen because they show that $(-2)^m$ is a real number for certain values of *m* and not for others. Notice that the values for *m* which result in $(-2)^m$ having no real solution (Ø) are those with decimals that, if written as fractions in lowest terms, would have even denominators. Notice also that the real values of $(-2)^m$ in the table alternate between positive and negative. This is because a negative base raised to an even power is positive (such as $(-2)^{-2} = 0.25$), and a negative base raised to an odd power is negative (such as $(-2)^{-3} = -0.125$).

m	$(-2)^m$	m	$(-2)^m$	m	$(-2)^m$	m	$(-2)^m$	m	$(-2)^m$
-3	-0.125	-2	0.25	-1	-0.5	0	1	1	-2
-2.9	Ø	-1.9	Ø	-0.9	Ø	0.1	Ø	1.1	Ø
-2.8	0.143...	-1.8	-0.287...	-0.8	0.574...	0.2	-1.148...	1.2	2.297...
-2.7	Ø	-1.7	Ø	-0.7	Ø	0.3	Ø	1.3	Ø
-2.6	-0.164...	-1.6	0.329...	-0.6	-0.659...	0.4	1.319...	1.4	-2.639...
-2.5	Ø	-1.5	Ø	-0.5	Ø	0.5	Ø	1.5	Ø
-2.4	0.189...	-1.4	-0.378...	-0.4	0.757...	0.6	-1.515...	1.6	3.031...
-2.3	Ø	-1.3	Ø	-0.3	Ø	0.7	Ø	1.7	Ø
-2.2	-0.217...	-1.2	0.435...	-0.2	-0.870...	0.8	1.741	1.8	-3.482...
-2.1	Ø	-1.1	Ø	-0.1	Ø	0.9	Ø	1.9	Ø

With *m* on the horizontal axis and $(-2)^m$ on the vertical, the result is:
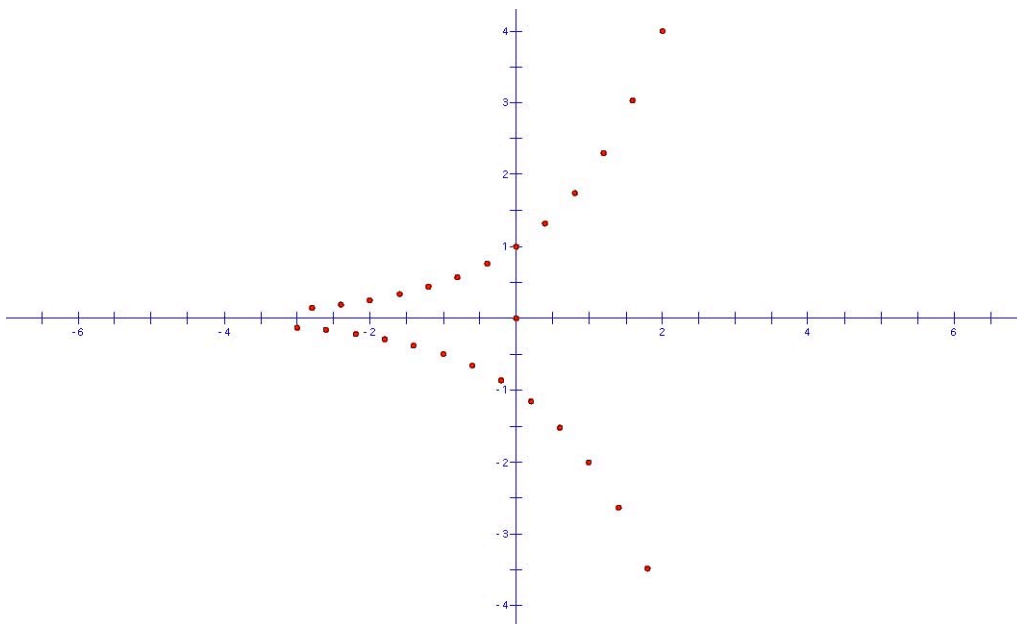
Mathematical Focus 3

The domain of the exponent, m, can determine the range of values for the base, x, of the exponential expression.

Thus far we have concentrated on how the value of the base, *x*, in x^m determines the domain of the exponent, *m*. But it is also true that if we want x^m to be defined in the real numbers, then the value of *m* puts certain limitations on *x*. (Note, however, that we are still limiting our discussion to real values, and subsets of the reals, of *x* and *m*.)

a. If *m* is a positive integer, then *x* may be any real number. For example, x^6 is always defined on the real numbers.

b. If *m* is a non-zero rational number with an odd denominator (when written in lowest terms), then *x* may be any real number. Note that $x^{\frac{3}{5}} = \sqrt[5]{x^3}$ is always defined on the real numbers.

c. If *m* is an integer and $m \leq 0$, then $x \in \Re, x \neq 0$. This is the contrapositive of a. in Focus 1.

d. If *m* is an irrational number or a nonzero rational number (in lowest terms) with an even denominator, then $x \geq 0$. (See previous discussion of even denominators.) This is the contrapositive of c. in Focus 1.