Situation 22: Operations with Matrices

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Prompt

Students in an Algebra II class had been discussing the addition of matrices and had worked on several examples of n x n matrices. Most were proficient in finding the sum of two matrices. Toward the end of the class period, the teacher announced that they were going to being working on the multiplication of matrices, and challenged the students to find the product of two 3 x 3 matrices:

$$\begin{bmatrix} 2 & 4 & 5 \\ 5 & 2 & 3 \\ 1 & 4 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \\ 5 & 2 & 3 \end{bmatrix}$$

Students began to work on the problem by multiplying each corresponding term in a way similar to how they had added terms. One student shared his work on the board getting a product of

$$\begin{bmatrix} 2 & 12 & 10 \\ 10 & 12 & 15 \\ 5 & 8 & 12 \end{bmatrix}$$

As the period ended, the teacher asked students to return to the next period with comments about the proposed method of multiplying and alternative proposals.

Commentary

The process of multiplying matrices is a frequently taught topic in secondary mathematics courses. However only the process is taught; the explanation of why process works is left out of the teaching of this concept. What makes the process of teaching matrix multiplication different was the inclusion of addition to the product of multiple entries. The three foci presented offer a variety of explanations of why addition is involved or why multiplication alone is not a sufficient means of multiplying two matrices. The first focus attempts to show why multiplication corresponding entries would not work for other known properties of matrices. The final two foci explain the inclusion of addition to matrix multiplication: the second focus shows a graphical understanding of multiplying matrices and the third focus attempts to justify the formula through a vector representation.

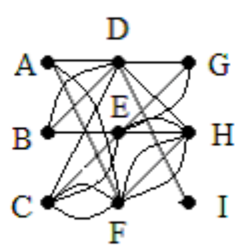
Mathematical Foci

Mathematical Focus 1

We can show the desired result through proof by contradiction. To begin, look at the identity matrix. By definition an identity matrix for multiplication is a square matrix made up of the main diagonal of ones and all other entries of zeroes. If we follow the teacher's approach to matrix multiplication, we can show it invalidates the definition of an identity matrix. First, the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ could be considered an identity matrix, since by the teacher's approach $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$. This would invalidate the definition of an identity matrix since $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is a 2x3 matrix, which is not a square matrix. Also by the teacher's approach, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ would be considered an identity since $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. However this matrix would invalidate the definition of an identity matrix since the entries not on the main diagonal are one, not zero. In order for the teacher's approach to work for matrix multiplication, it would destroy the notion of the identity matrix. Therefore, this approach could be not be usable for matrix multiplication.

Mathematical Focus 2

We can use the field of graph theory to develop the process of matrix multiplication. Let us come up with any graph of vertices and edges. Our vertices and edges are from {A, B, C} to {G, H, I} through {D, E, F}. For example, the graph



Now we can represent the connections between sets of vertices in matrix, using an adjacency matrix:

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$
$$\begin{array}{ccc} D & E & F \end{array}$$

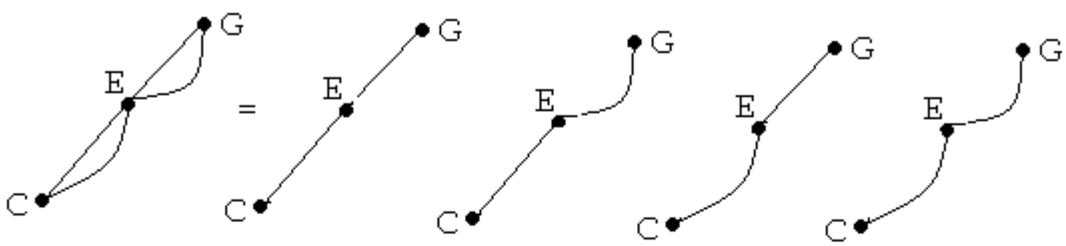
Each entry in the matrix represents the number of ways the two vertices are connected. For example, there are two ways to connect the vertices A and F, yet there is no way to connect the vertices A and E. In the same manner, we can create an adjacency matrix between {D, E, F} and {G, H, I}.

$$\begin{array}{c} D \\ E \\ F \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$
$$\begin{array}{ccc} G & H & I \end{array}$$

Now we want to find the number of ways between {A, B, C} and {G, H, I}. The only connection between these two sets is through the vertices {D, E, F}. To find the ways between {A, B, C} and {G, H, I}, find the number of connections to the individual vertices and then the total number of connections.

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 1 & 7 & 1 \\ 4 & 4 & 3 \\ 5 & 11 & 3 \end{bmatrix}$$
$$\begin{array}{ccc} G & H & I \end{array}$$

For example, in this product, there are five ways from vertex C to vertex G. There is the one path from C to D to G; there are four ways from C to E to G, as illustrated below:



Four is the product of the two ways from C to E and the two ways from E to G. Similarly, there are 11 ways from C to H: one from C to D to H, four from C to E to H (the product of two from C to E and two ways from E to H), and six from C to F to H (two ways from C to F and three ways from F to H).

The way we are finding the entries in the final matrix is to multiply all the different possibilities from the beginning to the intermediate points by the different possibilities from intermediate to destination. Finally, since there is more than one intermediate point, we add together the different possibilities.

Mathematical Focus 3

The process of matrix multiplication is similar to determining the dot product of two vectors. From two vector values the dot product yields a scalar quantity. Using the dot product of one row vector $\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$ and one column vector $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ gives the scalar value

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

This dot product value is the method by which the product entry is derived. For the i,jth entry in AB, one must use the ith row of A and the jth row of B.

Mathematical Focus 4

One of the ways students can visualize matrix multiplication is through a problem solving approach. Since matrices represent arrays of data, the process of matrix multiplication can be arrived at intuitively. This problem solving approach can be illustrated by the School Mathematics Study Group (1961, pgs. 24-26). In this approach, their textbook describes a manufacturer needing multiple parts to make different televisions, which can be written in matrix form:

$$\begin{array}{r} \text{Tubes} \\ \text{Speakers} \end{array} \underset{\text{Model A} \quad \text{Model B}}{\begin{bmatrix} 13 & 18 \\ 2 & 3 \end{bmatrix}}$$

Then the numbers of televisions sold in two different months are:

$$\begin{array}{r} \text{Model A} \\ \text{Model B} \end{array} \underset{\text{January} \quad \text{February}}{\begin{bmatrix} 12 & 6 \\ 24 & 12 \end{bmatrix}}$$

To find the total number of tubes and speakers sold during the two month period, it would make sense to determine the number of each part sold based on the number of models sold in each month. For example, the total number of tubes sold in January would equal the number of tubes for Model A times the number of televisions sold in January and the number of tubes sold for model B times the number of televisions sold in January: $13(12)+18(24)=156+432=588$. To figure out the remaining parts for each month, we would continue the same process:

$$\begin{array}{rc} \begin{array}{r}\text{Tubes}\\ \text{Speakers}\end{array} & \begin{bmatrix} 13(12)+18(24) & 13(6)+18(12) \\ 2(12)+3(24) & 2(6)+3(12) \end{bmatrix} \\ & \begin{array}{cc}\text{January} & \text{February}\end{array} \end{array}$$

$$\begin{array}{rc} \begin{array}{r}\text{Tubes}\\ \text{Speakers}\end{array} & \begin{bmatrix} 588 & 294 \\ 96 & 48 \end{bmatrix} \\ & \begin{array}{cc}\text{January} & \text{February}\end{array} \end{array}$$

References

Prompt: adapted from an observation at a Clarke County High School in February 2005.

Foci:

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