Situation 22: Operations with Matrices

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Prompt

Students in an Algebra II class had been discussing the addition of matrices and had worked on several examples of n x n matrices. Most were proficient in finding the sum of two matrices. Toward the end of the class period, the teacher announced that they were going to being working on the multiplication of matrices, and challenged the students to find the product of two 3 x 3 matrices:

$$\begin{bmatrix} 2 & 4 & 5 \\ 5 & 2 & 3 \\ 1 & 4 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \\ 5 & 2 & 3 \end{bmatrix}$$

Students began to work on the problem by multiplying each corresponding term in a way similar to how they had added terms. One student shared his work on the board getting a product of

$$\begin{bmatrix} 2 & 12 & 10 \\ 10 & 12 & 15 \\ 5 & 8 & 12 \end{bmatrix}$$

As the period ended, the teacher asked students to return to the next period with comments about the proposed method of multiplying and alternative proposals.

Commentary

The process of multiplying matrices is a frequently taught topic in secondary mathematics courses. However only the process is taught; the explanation of why process works is left out of the teaching of this concept. What makes the process of teaching matrix multiplication different was the inclusion of addition to the product of multiple entries. The three foci presented offer a variety of explanations of why addition is involved or why multiplication alone is not a sufficient means of multiplying two matrices. The first focus attempts to show why multiplication corresponding entries would not work for other known properties of matrices. The final two foci explain the inclusion of addition to matrix multiplication: the second focus shows a graphical understanding of multiplying matrices and the third focus attempts to justify the formula through a vector representation.

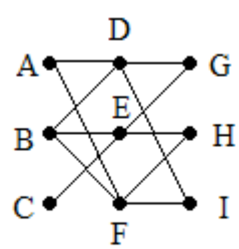
Mathematical Foci

Mathematical Focus 1

We can show the desired result through proof by contradiction. To begin, look at the identity matrix. By definition an identity matrix is a square matrix made up of the main diagonal of ones and all other entries of zeroes. If we follow the student teacher's approach to matrix multiplication, we can show it invalidates the definition of an identity matrix. First, the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ could be considered an identity matrix, since by the student teacher's approach $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$. This would invalidate the definition of an identity matrix since $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is a 2x3 matrix, which is not a square matrix. Also by the student teacher's approach, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ would be considered an identity since $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. However this matrix would invalidate the definition of an identity matrix since the entries not on the main diagonal are one, not zero. In order for the student teacher's approach to work for matrix multiplication, it would destroy the notion of the identity matrix. Therefore, this approach could be not be usable for matrix multiplication.

Mathematical Focus 2

We can visualize the product of two matrices as the number of paths of a graph of vertices and edges. For the graph



We can count the path for the set of vertices {A, B, C} to the set of vertices {D, E, F}. We can represent this information in matrix form

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\quad D \quad E \quad F$$

There could be a different matrix for the numbers of ways that the set of vertices {D, E, F} can be connected to {G, H, I}

$$\begin{array}{c} D \\ E \\ F \end{array}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\quad G \quad H \quad I$$

If we count the number of paths from {A, B, C} to {G, H, I} we have a third matrix.

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$
$$\quad G \quad H \quad I$$

If we count the number of paths from {A, B, C} to {G, H, I}, we must go through the vertices {D, E, F} individually and add all of the possibilities together. This addition of possibilities helps to explain why addition is involved in matrix multiplication.

Mathematical Focus 3

The process of matrix multiplication is similar to determining the dot product of two vectors. From two vector values the dot product yields a scalar quantity. Using the dot product of one row vector $\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$ and one column vector $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ gives the scalar value

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

This dot product value is the method by which the product entry is derived. For the i,jth entry in AB, one must use the ith row of A and the jth row of B.

References

Prompt: adapted from an observation at a Clarke County High School in February 2005.

Foci:

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