Situation 26: Absolute Value Equations

Prepared at University of Georgia
Center for Proficiency in Teaching Mathematics
6/28/05 – Kanita DuCloux

Edited at University of Georgia
6/21/06 -- Sarah Donaldson

Prompt

A student teacher begins a tenth-grade geometry lesson on solving absolute value equations by reviewing the meaning of absolute value with the class. They discussed one of the definitions of absolute value: that it represents a distance from zero on the number line and that the distance cannot be negative. He then asks the class what the absolute value tells you about the equation $|x| = 2$. To which a student responds "anything coming out of it must be 2". The student teacher states "*x* is the distance of 2 from 0 on the number line". Then on the board, the student teacher writes

$$
\begin{array}{lcl}
|x+2| = 4 & & \\
x+2 = 4 & \textit{and} & x+2 = -4 \\
x = 2 & & x = -6
\end{array}
$$

And graphs the solution on a number line. A puzzled student asks, "Why is it 4 and –4? How can you have –6? You said that you couldn't have a negative distance."

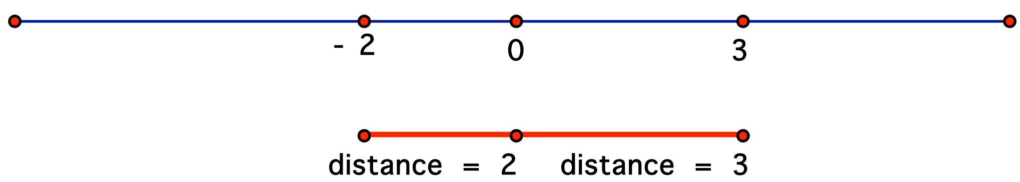
Commentary

In order to understand the concept of absolute value, a teacher must have a firm grasp of various definitions of it. Though not every definition must be mastered by the students, a teacher is better equipped to teach absolute value (or any concept, for that matter) the more thorough his/her own understanding is. Such mastery on the teacher's part will also help him/her address particular misunderstandings that students have about absolute value, such as the misunderstanding communicated in the questions of the student in this Prompt. The following foci address 4 different definitions of absolute value, and include various methods of solving the equation $|x + 2| = 4$.

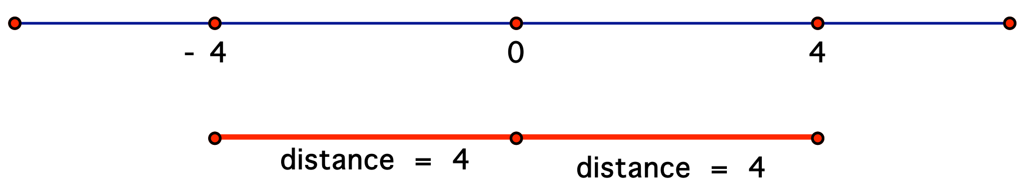
Mathematical Foci

Mathematical Focus 1: Absolute value as distance from zero on a number line.

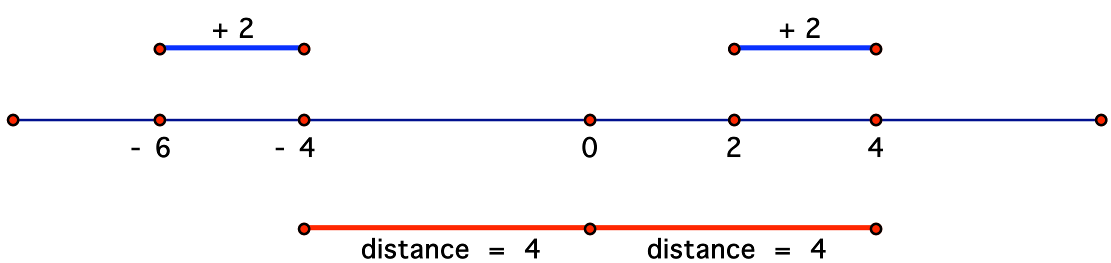
Absolute value can be defined as distance from zero on a number line. For example, $|3| = 3$ because 3 is 3 units away from zero, and $|-2| = 2$ because -2 is 2 units from zero.



$|x| = 4$ means that the distance from x to zero on a number line is 4. On the number line below, it can be seen that there are two points that are 4 units away from zero: 4 and -4.



In $x + 2 = 4$, x represents a number such that, when 2 is added to it, the result is 4. In $|x + 2| = 4$, x represents a number such that, when 2 is added to is, the result will be 4 units away from zero on a number line. When the points -6 and 2 are shifted 2 units to the right (ie 2 is added to each of them), the result is two points (-4 and 4) each lying 4 units away from zero. Therefore $x = -6$ and $x = 2$ are solutions to $|x + 2| = 4$.



There is an algebraic method to solve the equation $|x + 2| = 4$ using the definition of absolute value as distance from zero on a number line. Absolute value equations often yield more than one solution. In $|x| = 4$, for example, there are two values for x that make the equation true, 4 and -4, because both $|4|$ and $|-4|$ are 4. That is, both 4 and -4 are 4 units away from zero on a number line.

Expanding this notion to other absolute value equations, such as $|x + 2| = 4$, there will again be two possible solutions for x. To get these solutions, all possibilities for the value of $(x + 2)$ must be listed:

$x + 2 = 4$ and $x + 2 = -4$

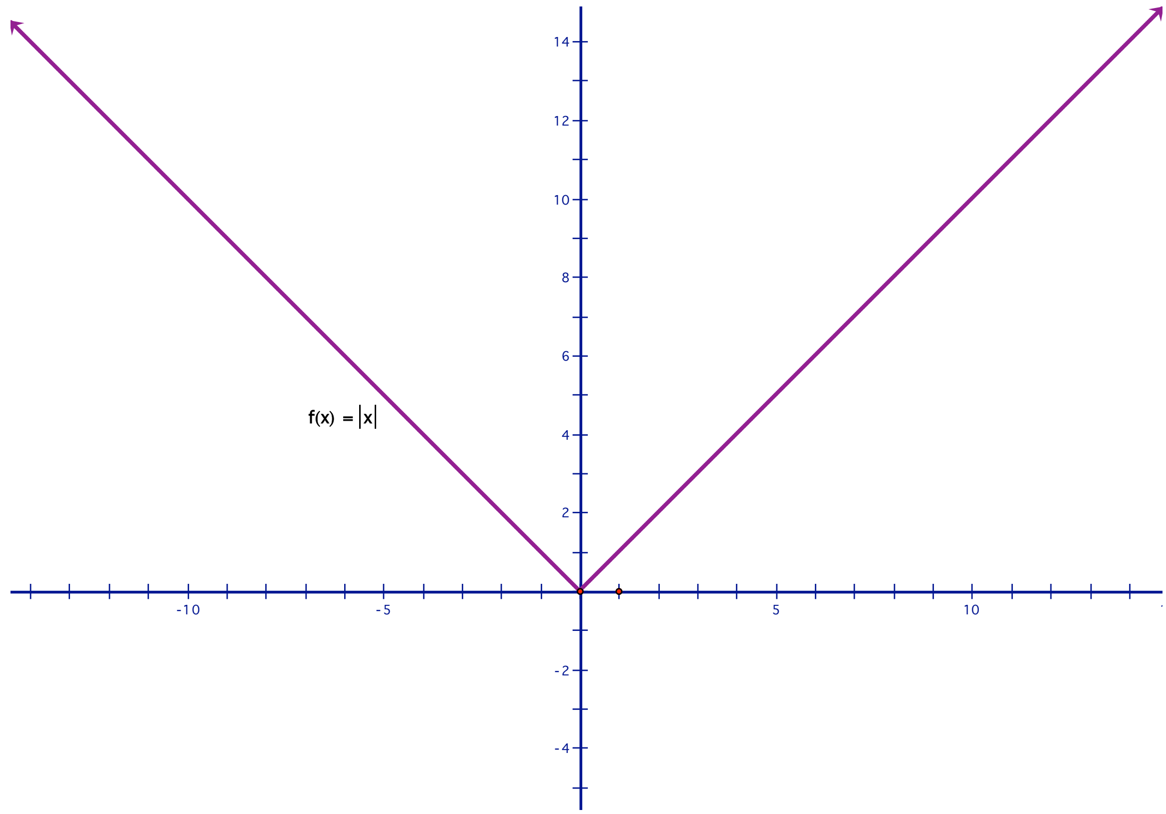
$x = 2$ and $x = -6$.

Each solution can be checked in the original equation to see that they make the equation true:

Does $\lvert(2) + 2\rvert = 4$?	Does $\lvert(-6) + 2\rvert = 4$?
$\lvert 4\rvert = 4$?	$\lvert -4\rvert = 4$?
YES	YES

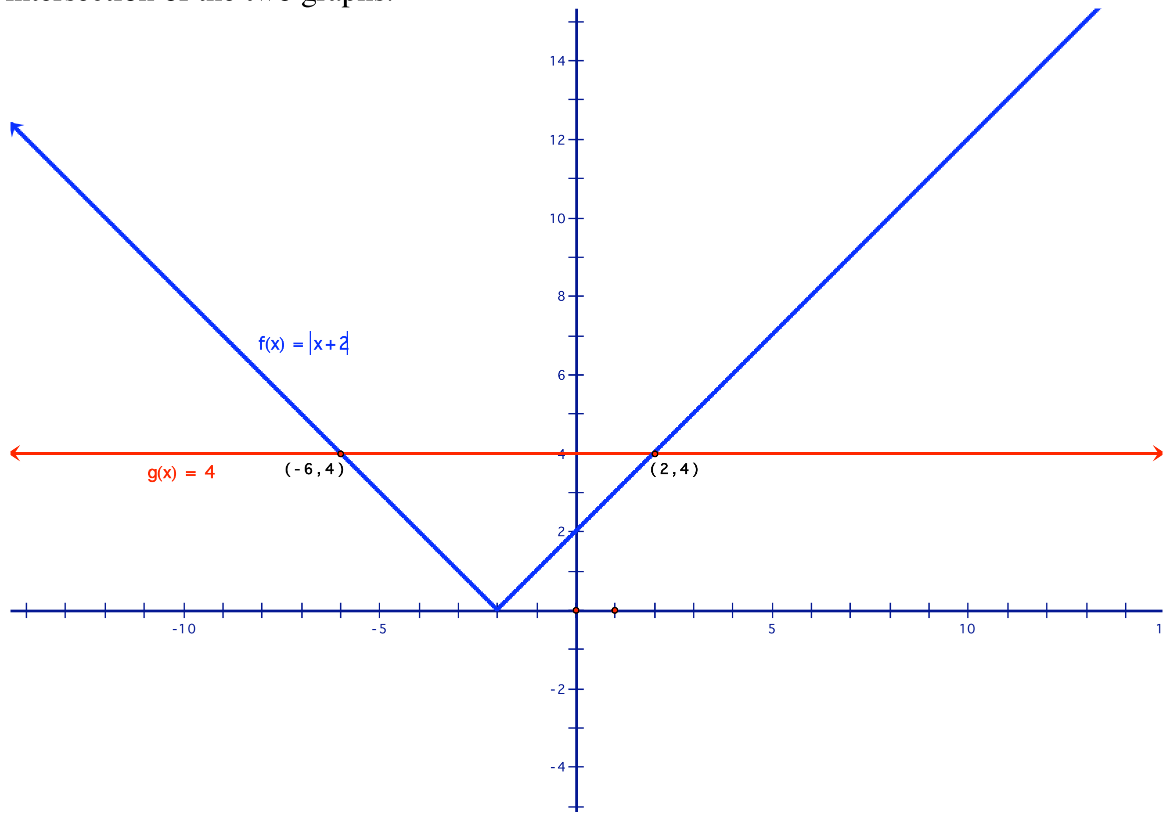
Mathematical Focus 2: Absolute value as a function

Absolute value can be defined as the function $f(x) = |x|$ whose graph is:



It can be seen that the domain of the function is x = all real numbers, and the range is $y \geq 0$. Here is a visual representation of "absolute value is never negative." The y-value (range) is never negative (the function doesn't exist below the x-axis), but the x-value (domain) could be any real number, positive or negative. Using absolute value notation $y = |x|$, this idea of domain and range means that what is inside the absolute value symbols (x) **can** be negative while the absolute value itself (y) **cannot** be negative.

The graph of the absolute value function can be used to solve $|x + 2| = 4$ by examining the graphs of $f(x) = |x + 2|$ and $g(x) = 4$ to see where they intersect. $|x + 2| = 4$ at the point of intersection of the two graphs:



The graphs intersect at $x = -6$ and $x = 2$, which means that $x = -6$ and $x = 2$ are solutions of the equation $|x + 2| = 4$.

Mathematical Focus 3: Absolute value of x defined separately for $x \geq 0$ and $x < 0$

A third definition of absolute value is: $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

Using this definition to solve the equation $|x + 2| = 4$, one would set up two equations:

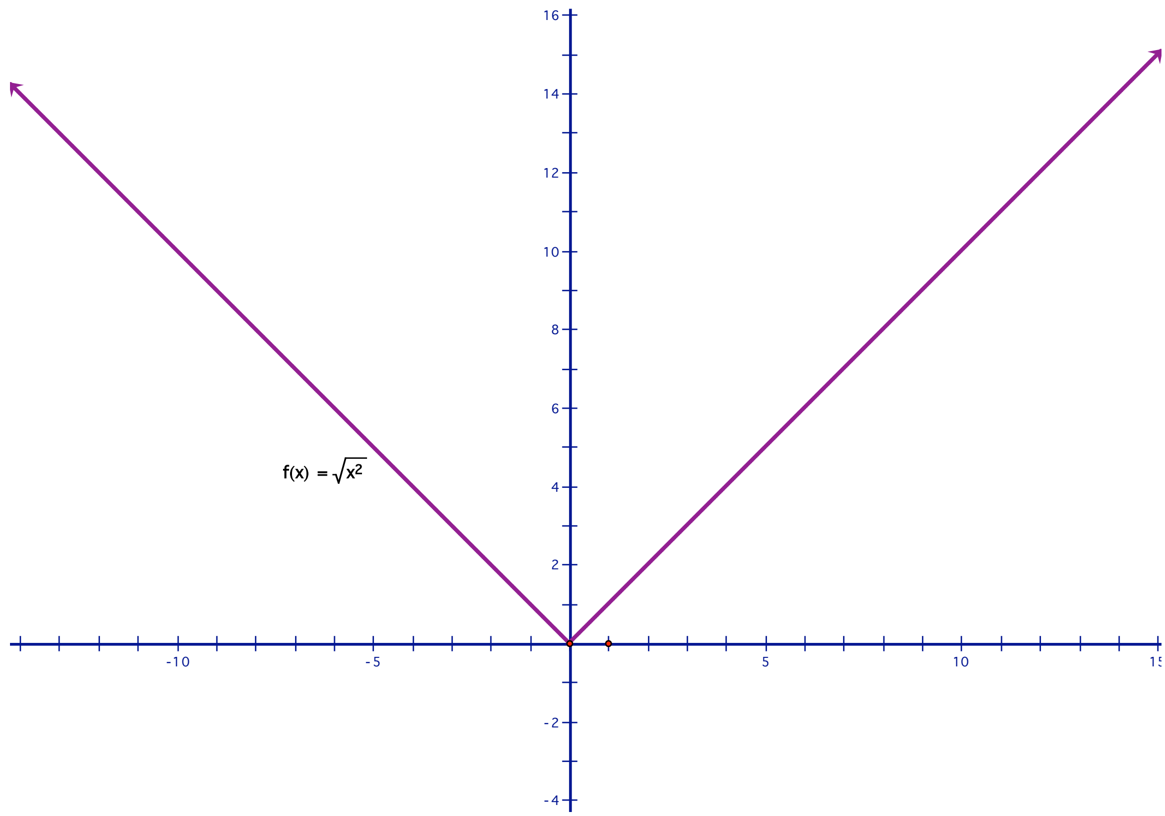
$\lvert x + 2\rvert = x + 2$ if $x + 2 \geq 0$	$\lvert x + 2\rvert = -(x + 2)$ if $x + 2$ is < 0
$4 = x + 2$ $2 = x$ $x = 2$	$4 = -(x + 2)$ $4 = -x - 2$ $6 = -x$ $x = -6$

So the solutions are $x = 2$ and $x = -6$.

Mathematical Focus 4: Absolute value of x as the positive square root of x^2

A fourth definition of absolute value is: $|x| = +\sqrt{x^2}$

This is the graph of the function $f(x) = \sqrt{x^2}$



Algebraically, this definition can be used to solve $|x + 2| = 4$,

$$
\begin{aligned}
&|x+2| = 4 \\
&+\sqrt{(x+2)^2} = 4 \\
&(x+2)^2 = 16 \\
&x^2 + 4x + 4 = 16 \\
&x^2 + 4x - 12 = 0 \\
&(x+6)(x-2) = 0 \\
&x = -6, x = 2
\end{aligned}
$$

Mathematical Focus 5: Table of values

Make a table of various inputs (x-values) and outputs ($|x + 2|$) and see which inputs produce an output of 4.

x	$\lvert x + 2\rvert$
-7	$\lvert(-7) + 2\rvert = \lvert-5\rvert = 5$
-6	$\mathbf{\lvert(-6) + 2\rvert = \lvert-4\rvert = 4}$
-5	$\lvert(-5) + 2\rvert = \lvert-3\rvert = 3$
-4	$\lvert(-4) + 2\rvert = \lvert-2\rvert = 2$
-3	$\lvert(-3) + 2\rvert = \lvert-1\rvert = 1$
-2	$\lvert(-2) + 2\rvert = \lvert0\rvert = 0$
-1	$\lvert(-1) + 2\rvert = \lvert1\rvert = 1$
0	$\lvert(0) + 2\rvert = \lvert2\rvert = 2$
1	$\lvert(1) + 2\rvert = \lvert3\rvert = 3$
2	$\mathbf{\lvert(2) + 2\rvert = \lvert4\rvert = 4}$
3	$\lvert(3) + 2\rvert = \lvert5\rvert = 5$

The input values which result in an output of 4 are $x = -6$ and $x = 2$.