Situation 26: Absolute Value

Prepared at University of Georgia
Center for Proficiency in Teaching Mathematics
6/28/05 – Kanita DuCloux
Additions by Sarah Donaldson 6/16/06

Prompt

A student teacher begins a tenth-grade geometry lesson on solving absolute value equations by reviewing the meaning of absolute value with the class. They discussed that the absolute value represents a distance from zero on the number line and that the distance cannot be negative. He then asks the class what the absolute value tells you about the equation $|x|=2$. To which a male student responds "anything coming out of it must be 2". The student teacher states "*x* is the distance of 2 from 0 on the number line". Then on the board, the student teacher writes

$$|x+2|=4$$
$$x+2=4 \quad \textit{and} \quad x+2=-4$$
$$x=2 \qquad\qquad x=-6$$

And graphs the solution on a number line. A puzzled female student asks, "Why is it 4 and –4? How can you have –6? You said that you couldn't have a negative distance?"

How do you respond to the student's questions?

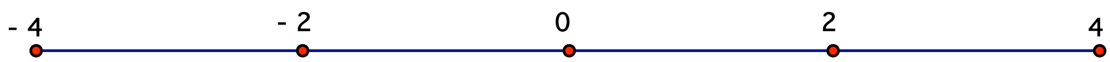
Commentary

In order to understand the concept of absolute value, a teacher must have a firm grasp of various definitions of it. Though not every definition must be mastered by the students, a teacher is better equipped to teach absolute value (or any concept, for that matter) the more thorough his/her own understanding is. Such mastery on the teacher's part will also help him/her address particular misunderstandings that students have about absolute value, such as the misunderstanding communicated in the questions of the student in this Prompt. The following foci address 4 different definitions of absolute value, and include various methods of solving the equation $|x + 2| = 4$.

Mathematical Foci

Mathematical Focus 1: Absolute value as distance from zero on a number line.

Absolute value can be defined as distance from zero on a number line. For example, $|3| = 3$ because 3 is 3 units away from zero, and $|-2| = 2$ because -2 is 2 units from zero.



$|x| = 4$ means that the distance from x to zero on a number line is 4. On the number line below, it can be seen that there are two points that are 4 units away from zero: 4 and -4.

In $x + 2 = 4$, x represents a number such that, when 2 is added to it, the result is 4. In $|x + 2| = 4$, x represents a number such that, when 2 is added to is, the result will be 4 units away from zero on a number line. When the points -6 and 2 are shifted 2 units to the right (ie 2 is added to each of them), the result is two points (-4 and 4) each lying 4 units away from zero. Therefore $x = -6$ and $x = 2$ are solutions to $|x + 2| = 4$.

There is an algebraic method to solve the equation $|x + 2| = 4$ using the definition of absolute value as distance from zero on a number line.

When solving for an unknown (such as "x") in an equation, one must list all the possible real solutions (values for x) that make the equation true. Some equations yield only one real solution. For example, in $x + 2 = 5$, the only real number that x could be to make the equation true is 3. Other equations yield more than one solution. For example, if $x^2 = 9$, then x could be either 3 or -3 because both $(3)^2$ and $(-3)^2$ equal 9.

Absolute value equations often yield more than one solution. In $|x| = 4$, for example, there are two values for x that make the equation true, 4 and -4, because both $|4|$ and $|-4|$ are 4. That is, both 4 and -4 are 4 units away from zero on a number line (see number line above).

Expanding this notion to other absolute value equations, such as $|x + 2| = 4$, there will again be two possible solutions for x. To get these solutions, all possibilities for the value of $(x + 2)$ must be listed. $(x + 2)$ could be 4 or -4 because (as was already stated) both $|4|$ and $|-4|$ equal 4. Listing each of these possibilities as equations, then,

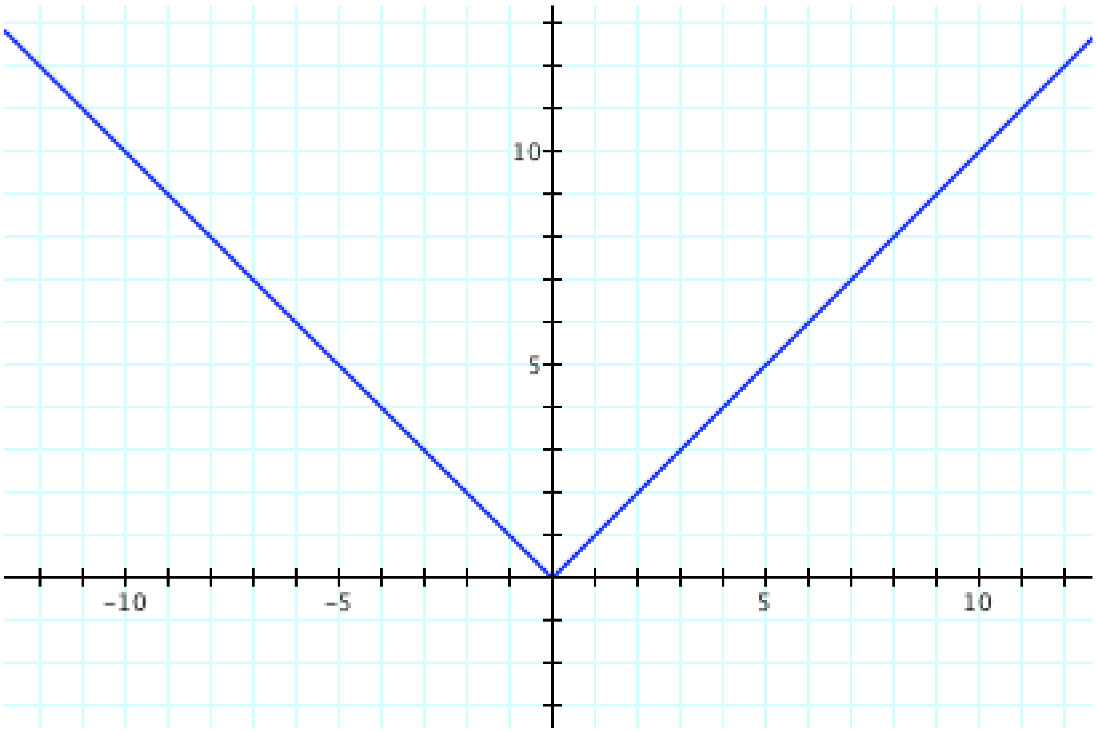
$$x + 2 = 4 \qquad \text{and} \qquad x + 2 = -4$$

If $x + 2 = 4$, then $x = 2$, and if $x + 2 = -4$, then $x = -6$. Each solution can be checked in the original equation to see that they make the equation true:

Does $|(2) + 2| = 4$?	Does $|(-6) + 2| = 4$?
$|4| = 4$?	$|-4| = 4$?
YES	YES

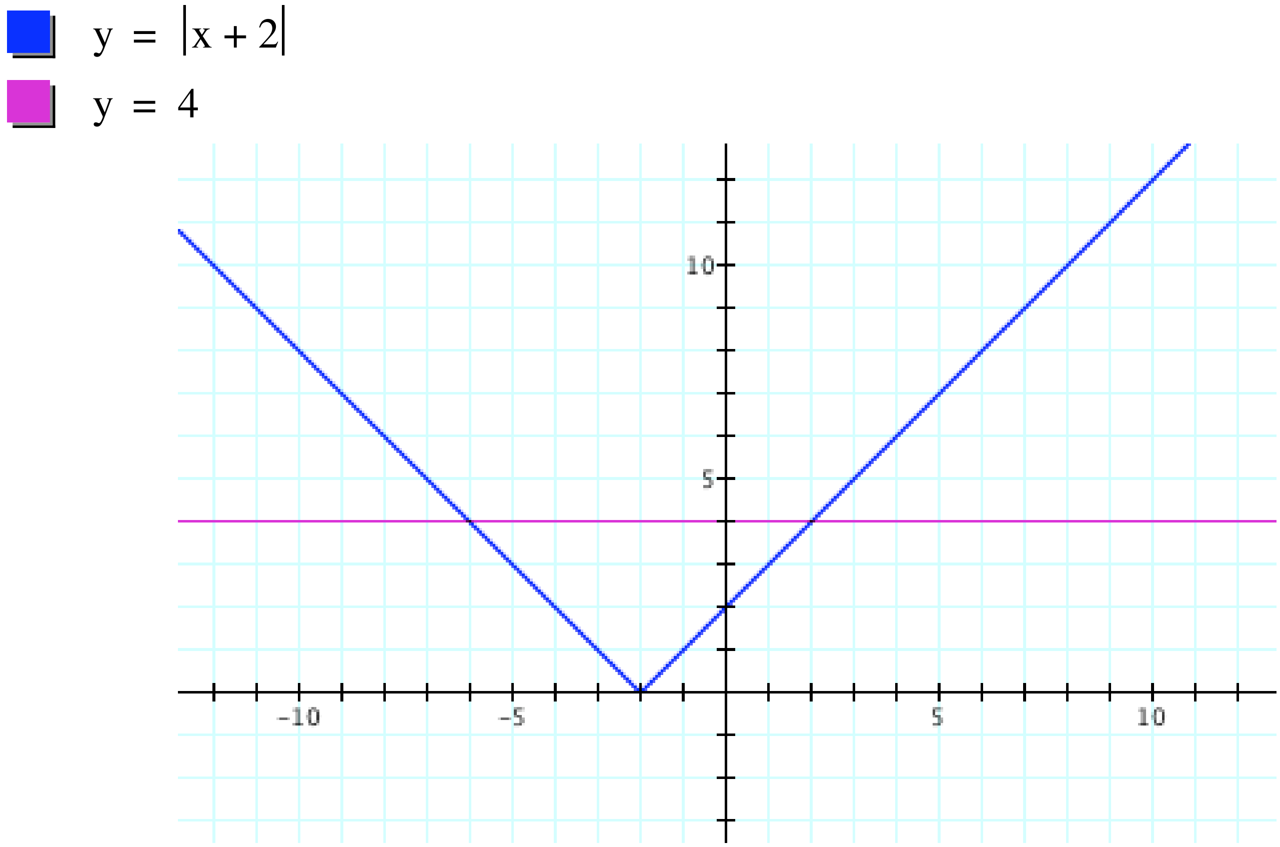
Mathematical Focus 2: Absolute value as a function

Absolute value can be defined as the function $f(x) = |x|$ whose graph is:



It can be seen that the domain of the function is x = all real numbers, and the range is $y \geq 0$. Here is a visual representation of "absolute value is never negative." The y-value (range) is never negative (the function does not exist below the x-axis), but the x-value (domain) could be any real number, positive or negative. Using absolute value notation $y = |x|$, this idea of domain and range means that what is inside the absolute value symbols (x) **can** be negative while the absolute value itself (y) **cannot** be negative.

The graph of the absolute value function can be used to solve the equation $|x + 2| = 4$ by examining the graphs of $y = |x + 2|$ and $y = 4$ to see where they intersect.



$|x + 2| = 4$ at the point of intersection of the two graphs. The graphs intersect at $x = -6$ and $x = 2$, which means that $x = -6$ and $x = 2$ are solutions of the equation $|x + 2| = 4$.

Mathematical Focus 3**:** Absolute value of x defined separately for $x \geq 0$ and $x < 0$

A third definition of absolute value is: $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

Using this definition to solve the equation $|x + 2| = 4$, one would set up two equations:

$|x + 2| = x + 2$ if $x + 2 \geq 0$

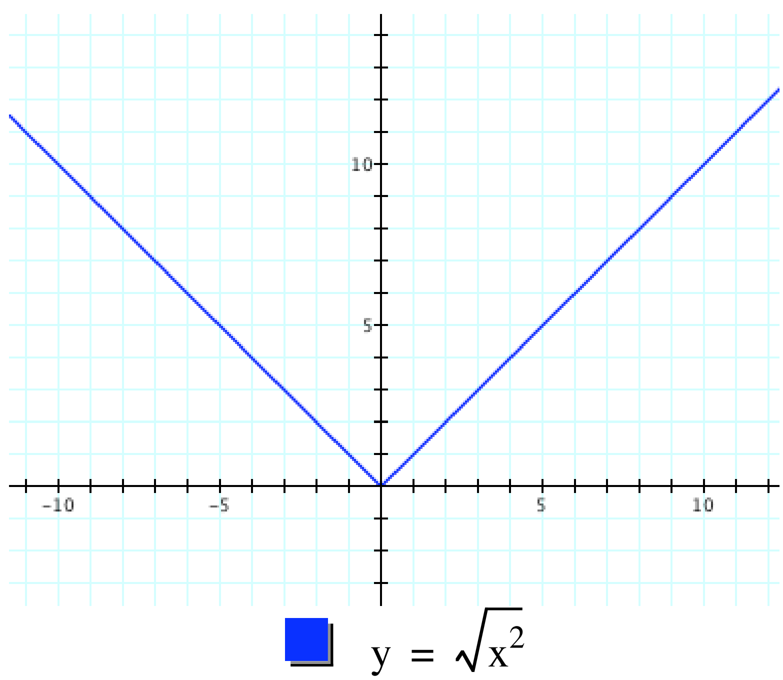
$4 = x + 2$
$2 = x$
$x = 2$

$|x + 2| = -(x + 2)$ if $x + 2$ is < 0

$4 = -(x + 2)$
$4 = -x - 2$
$6 = -x$
$x = -6$

So the solutions are $x = 2$ and $x = -6$.

Mathematical Focus 4: Absolute value of x as the positive square root of x^2



Mathematical Focus 5: making a chart of values

Make a table of various inputs (x-values) and outputs ($|x + 2|$) and see which inputs produce an output of 4.

x	$\|x + 2\|$
-8	$\|(-8) + 2\| = \|-6\| = 6$
-7	$\|(-7) + 2\| = \|-5\| = 5$
-6	$\mathbf{\|(-6) + 2\| = \|-4\| = 4}$
-5	$\|(-5) + 2\| = \|-3\| = 3$
-4	$\|(-4) + 2\| = \|-2\| = 2$
-3	$\|(-3) + 2\| = \|-1\| = 1$
-2	$\|(-2) + 2\| = \|0\| = 0$
-1	$\|(-1) + 2\| = \|1\| = 1$
0	$\|(0) + 2\| = \|2\| = 2$
1	$\|(1) + 2\| = \|3\| = 3$
2	$\mathbf{\|(2) + 2\| = \|4\| = 4}$
3	$\|(3) + 2\| = \|5\| = 5$
4	$\|(4) + 2\| = \|6\| = 6$
5	$\|(5) + 2\| = \|7\| = 7$
6	$\|(6) + 2\| = \|8\| = 8$

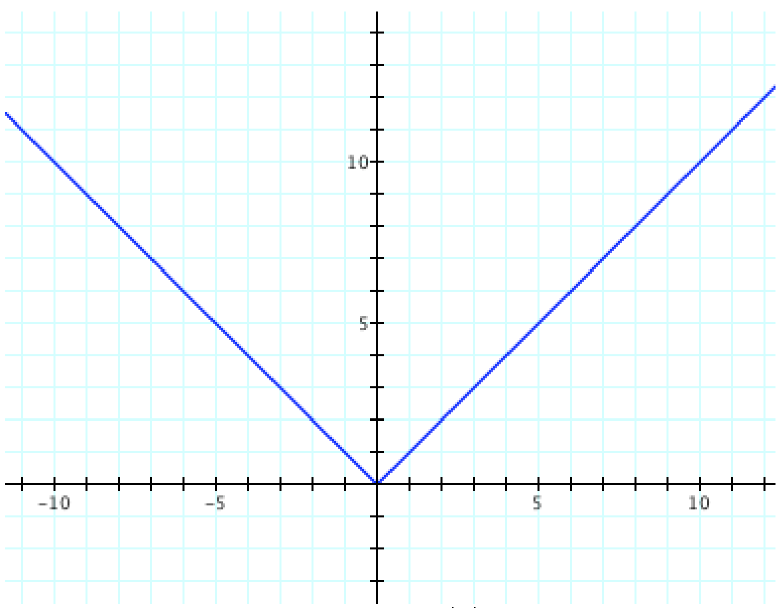
Commentary

The student's questions, "Why is it 4 and –4? How can you have –6? You said that you couldn't have a negative distance?" communicate her difficulty understanding that a solution of an absolute value equation could be negative (ie -6 is a solution of $|x + 2| = 4$). This is likely because it has been ingrained (improperly) that a negative result is never possible when dealing with absolute value. This is a misunderstanding of the notion that "absolute value can never be negative."

A correct explanation of "absolute value can never be negative" is necessary. Such an explanation might include a discussion of the distinction between a negative solution (value for x) and an absolute value being negative. For example, the above problem ($|x + 2| = 4$) yields a negative solution ($x = -6$), however the absolute value itself ($|x + 2|$) remains positive (+4).

It may be helpful to show the kind of equation for which there is no solution. For example, $|x + 2| = -4$ cannot be solved because the equation states that an absolute value is negative (which is impossible).

Graphically, the fact that <u>absolute value is never negative, but x can be negative</u> can be seen by noting that the <u>domain</u> of the graph of $y = |x|$ is x = all real numbers (could be positive or negative), while the <u>range</u> of $y = |x|$ is $y \geq 0$ (never negative). In other words, the graph of $y = |x|$ extends infinitely to the <u>left</u> and <u>right</u> of the <u>y-axis</u>, but does not exist <u>below</u> the <u>x-axis</u>:



$y = |x|$

That is, in $y = |x|$, x can be negative but y cannot. In the equation $|x + 2| = 4$, x and/or x + 2 can be negative, but 4 cannot.

Simplifying further (considering absolute value with only numbers and no letters), what's <u>inside</u> the absolute value can be negative, but what's <u>outside</u> cannot.

Examples:

$|-3| = 3$

$|-7| = 7$

<u>can</u> be negative   <u>never</u> negative