MAC-CPTM Situations Project

Situation 28: Adding Square Roots

(Includes material from Situation 32 Radicals)

Situation 28 Prepared at the University of Georgia
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Prompt

A mathematics teacher, Mr. Fernandez, is bothered by his ninth grade algebra students' responses to a recent quiz on radicals, specifically a question about square roots in which the students added $\sqrt{2}$ and $\sqrt{3}$ and got $\sqrt{5}$.

Commentary

There are many ways to think about the operations regarding square roots, and this situation will attempt to show several ways of thinking about adding them. Focus 1 will use a geometric approach to show how to help students develop a better sense of square roots as numbers. The reason for Focus 1—square roots of whole numbers—in response to this prompt is primarily because this approach offers students the possibility of building up their knowledge of square roots as lengths (of sides of squares and as the hypotenuse of a right triangle) so as to consider what is possible and not possible in combining and operating with these relatively "new" (for ninth graders) mathematical objects.

Generating the notion of square roots as mathematical objects in relation to other more familiar mathematical objects like whole numbers and fractions, is crucial to constructing square roots as numbers, let alone combining and operating with them. Part of the reason that students will add

$$\sqrt{2} + \sqrt{3}$$

and get

$$\sqrt{5}$$

has to do with their familiarity with whole numbers as mathematical objects and their familiarity with combining and operating with whole numbers. Thus, a major issue to address in this situation is why those intuitions (built up over years!) do not hold. Note that students hardly have the same amount of time to build up intuitions about combining and operating with square roots. So this situation continues the development of this quantitative approach toward that end.

Focus 2 will use algebra to prove that the statement $\sqrt{2} + \sqrt{3} = \sqrt{5}$ is false. (This focus relies on the fact that $(a+b)^2 \neq a^2 + b^2$. For a more in depth discussion of this, see Situation 37). Focus 3 uses properties of functions, specifically when f(a + b) = f(a) + f(b).

It is important to notice that Foci 2 and 3 are ones for which the arguments can be generalized to other radicals. In Focus 2, the algebraic argument can be generalized by working with $\left(\sqrt[n]{a} + \sqrt[n]{b}\right)^n$ and comparing this expression with $\left(\sqrt[n]{a+b}\right)^n$. Furthermore, Focus 3, since $f(x) = \sqrt[n]{x}$ is an increasing function, the argument presented is easily generalized. However, Focus 1 cannot be generalized for other radicals. This geometric approach is unique for square roots, but still gives a powerful way to develop the reasoning behind operations on square roots.

Mathematical Foci

Mathematical Focus 1—roots of whole numbers

By using a quantitative approach we can focus primarily on square roots as lengths and use geometrical problem solving with only some numeric or algebraic calculation. Use of a tool like Geometer's Sketchpad will be helpful in this regard, although some of what's mentioned below can occur without GSP.

Starting with a square of area 1 square unit (see Figure 1), can you make a square of area 2 square units?
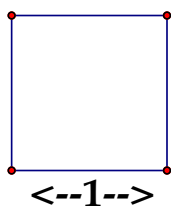
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Figure 1

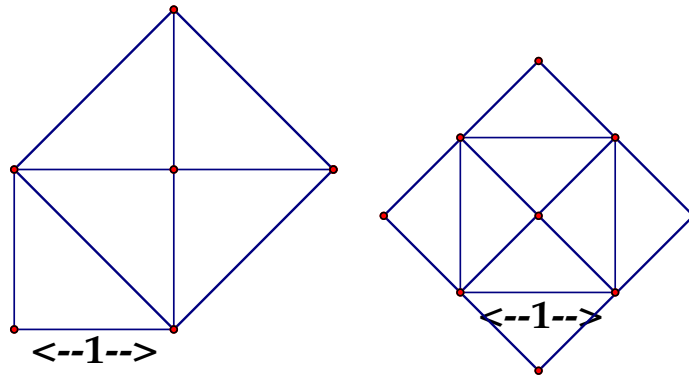This problem can be solved in multiple ways, see Figure 2 below.

Figure 2

If a square with area 1 square unit has side root 1, which is 1, then a square with area 2 has side $\sqrt{2}$.

Using the technique for making a square of area 2, (e.g., drawing diagonals and using the isosceles right triangles that are formed, circumscribing squares, etc.), we can then think about making squares of other areas, such as area 3 or of area 5. An example of such a construction can be seen in Figure 3 below and **page 1** of the GSP sketch **Adding Radicals.gsp**).
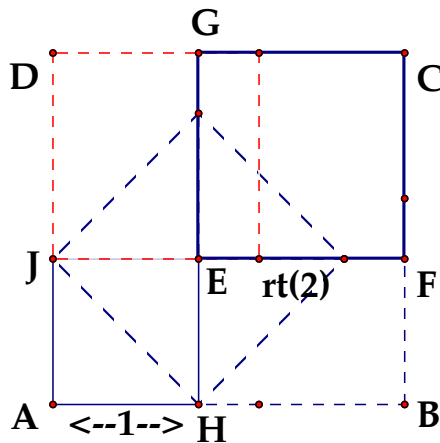
Note: to move the square of area 2 so that it touches the square with area 1 at one point as shown, the blue dashed square was rotated 45 degrees to make the red dashed square and then translated 1 unit to the right. Together, the areas of AHEJ and EFCG have area 3 square units.

Figure 3

This sketch shows one way to "combine" a square of area 1 (AHEJ) and a square of area 2 (EFCG). Note that using this diagram to make a square of area 3 is an example of the Pythagorean Theorem.

The biggest square, ABCD, consists of an area of 1 square unit, an area of 2 square units, and two rectangles that are each 1 unit by $\sqrt{2}$ units. When the areas of the two rectangles are cut apart into 4 right triangular areas (with legs 1 and root 2 units) and separated, the remaining area in ABCD should also have area 1 + 2 = 3 (see Figure 5).
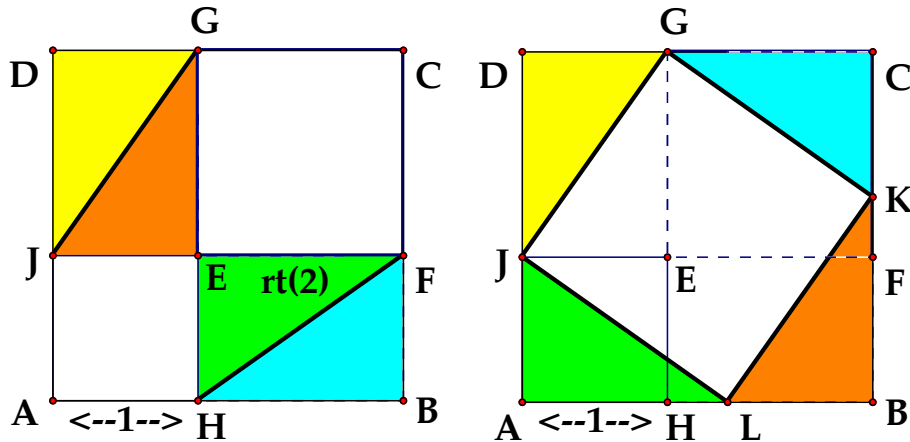
Figure 4

It remains to justify that this remaining area (GKLJ in Figure 4) is a square, which can be done using the 4 congruent right triangles (i.e., GKLJ is at least a rhombus and then it must have at least one right angle because, for example, angles AJL and DJG are complementary). So the sides of square GKLJ can be said to be

$$\sqrt{3}$$

Using this technique, we can now make a square with area 5. Note that here the Pythagorean Theorem is used directly as a theorem about areas to make these area relationships and resulting lengths $\sqrt{5}$.

Locating all of these lengths on a number line in order to "fill in" the number line can help to develop a sense of the relationship of these square roots to numbers like the whole numbers and fractions (see **pages 2, 3, and 4** of the GSP sketch **Adding Radicals.gsp**).

By using these constructions, the relative size of each square root expression and the reasonableness of the equation $\sqrt{2} + \sqrt{3} = \sqrt{5}$ becomes more clear. In particular, once $\sqrt{2}, \sqrt{3},$ and $\sqrt{5}$ are constructed, the lengths can be compared in a variety of ways to show the inconsistency with the equation above (see **page 5** Adding Radicals.gsp).

One of the underlying issues in the situation is why intuitions about addition, based on addition of whole numbers, do *not* hold with fractions or square roots, while intuitions about multiplication (again based on multiplication of whole numbers) *do*. (Note that one reason for this issue is that students assume that "square rooting" distributes across addition, i.e., the erroneous concept that

$$\sqrt{2 + 3} = \sqrt{2} + \sqrt{3}.$$

Although this kind of conception can be refuted using counterexamples, this will not be included so that this discussion will develop a quantitative approach toward this issue). So a major reason for taking a quantitative approach to work with square roots as outlined above is to investigate this issue.

After square roots have been constructed as the lengths of sides of squares, questions can arise about how to combine these lengths. Lengths can certainly be added by joining them contiguously, but, since $\sqrt{2}$ and $\sqrt{3}$ cannot be written as whole numbers or fractions, we have no way to know whether we can notate their combination with a single graphic item, the way we can combine 2.5 and 3.75 into 6.25. So (at least for the moment),

$$\sqrt{2} + \sqrt{3}$$

is exactly that,

$$\sqrt{2} + \sqrt{3}$$

Furthermore, using lengths it is possible to develop intuitions about

$$\sqrt{2} + \sqrt{3}$$

NOT being equal to

$$\sqrt{5}$$

(see page 5 of **Adding Radicals.gsp**).

Mathematical Focus 2

To determine if the equation $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$ is a true statement, one can use algebraic techniques to simply each side of the equation. To begin, we can square each side of the equation. When the left-hand side is squared, one must use the distribution property as well as the commutative and associative properties of addition to simply.

$$\left(\sqrt{2} + \sqrt{3}\right)^2 =$$
$$\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{2} + \sqrt{3}\right) =$$
$$\left(\sqrt{2} + \sqrt{3}\right)\sqrt{2} + \left(\sqrt{2} + \sqrt{3}\right)\sqrt{3} =$$
$$\left(\sqrt{2} \cdot \sqrt{2} + \sqrt{3} \cdot \sqrt{2}\right) + \left(\sqrt{2} \cdot \sqrt{3} + \sqrt{3} \cdot \sqrt{3}\right) =$$
$$\left(\sqrt{4} + \sqrt{6}\right) + \left(\sqrt{6} + \sqrt{9}\right) =$$
$$2 + \sqrt{6} + \sqrt{6} + 3 =$$
$$5 + 2\sqrt{6}$$

When the right-hand side is squared, however, it is immediately apparent that the sentence cannot be true. Namely, $\left(\sqrt{2+3}\right)^2 = 2 + 3 = 5$.

However, such an approach relies on the fact that the students will understand that $5 + 2\sqrt{6} \neq 5$. One approach to this would be to use a geometric representation of the numbers as areas of squares. The first square, representing $\left(\sqrt{2} + \sqrt{3}\right)^2$ has four regions. Two squares of areas 2 and 3 as well as two rectangles with area $\sqrt{6}$. However, the square representing $\left(\sqrt{5}\right)^2$ only consists of one region of area 5.
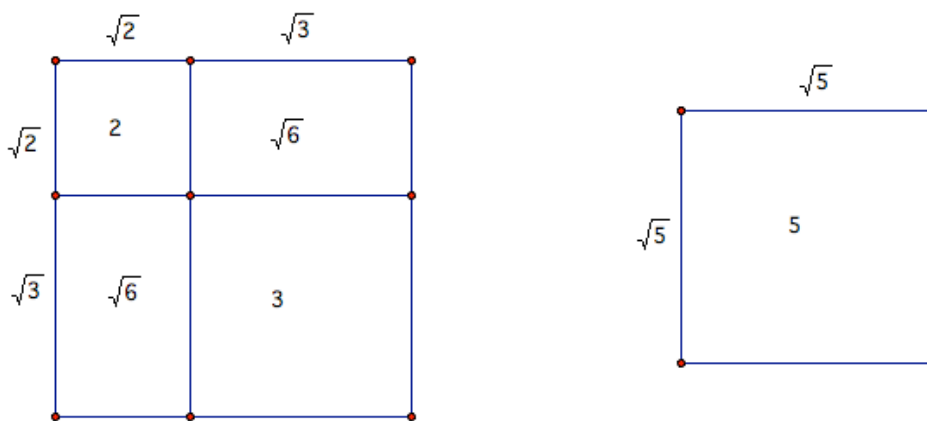
(For an algebraic discussion, see Situation 37.)

Mathematical Focus 3

Another way to approach this problem is to consider the properties of the function $f(x) = \sqrt{x}$. That is, we want to consider the property that $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$. In function notation, we would be considering when $f(2+3) = f(2) + f(3)$. In general, this becomes the question to whether the following property holds for the square root function: $f(a+b) = f(a) + f(b)$. If a function has this property, then it is called a homomorphism (in abstract algebra terms) or a linear transformation (in linear algebra terms).

Very few functions in fact have this property with the most common being linear functions representing a direct variation relationship between variables. For example, if $f(x) = kx$, then $f(a+b) = k(a+b) = ka + kb = f(a) + f(b)$. On the other hand, this property fails for most other functions in the high school curriculum; polynomial functions of degree larger than 1, functions that include fractional powers, and all of the trigonometric functions.

Specifically, one can look at the following example: $f(5) = f(4+1) = f(4) + f(1) = 1 + 2 = 3$. If this statement was true, $\sqrt{5} = 3$. However, since $4 < 5 < 9$ and $f(x) = \sqrt{x}$ is a strictly increasing function, $\sqrt{4} < \sqrt{5} < \sqrt{9}$. Therefore, since $\sqrt{4} = 2$ and $\sqrt{9} = 3$, $2 < \sqrt{5} < 3$. This implies that $\sqrt{5} \neq 3$.