Situation 32: Radicals
Prepared at UGA
Center for Proficiency in Teaching Mathematics
9/29/05—Amy Hackenberg

DISCUSSION

Prompt

A mathematics teacher, Mr. Fernandez, is bothered by his ninth grade algebra students' responses to a recent quiz on radicals—simplifying fractions with radicals in the denominator, adding radicals, and multiplying radicals. Some of them got mostly right answers but many students added root 2 and root 3 and got root 5, and others simplified the reciprocal of root 3 to root 3 divided by 9. Mr. Fernandez is disturbed over the incorrect answers but even more disturbing to him is the sense that none of this work is meaningful for his students—even for the ones who know the rules.

What mathematical knowledge does Mr. Fernandez need so that he might change his approach to radicals with his students?

[Comment: Is this vignette about addressing the pedagogical issue of developing more meaningfulness about radicals rather than an issue about what mathematical knowledge should a teacher draw on to address the mathematical issue at hand?]

Response: In my opinion, mathematical knowledge for teaching includes knowing how to use your mathematical knowledge in communication with others who don't necessarily have similar mathematical knowledge as yours. That may mean that my conception of mathematical knowledge for teaching includes a pedagogical aspect, as you say. However, for me it seems odd *not* to include this kind of knowledge in mathematical knowledge for teaching, as your question implies. That is, from my point of view, the mathematical knowledge of a teacher is directly involved in posing problems and activities for students so that they might generate their own meaningful mathematical knowledge.

Mathematical Foci[1]

[Comment: This is different from the others: one math topic (radicals) leads to four quite different mathematical foci not four different foci addressing the same mathematical issue.]

Response: I agree. It's possible that the Situations Project does not want to include this kind of approach.

Mathematical Focus 1—roots of whole numbers

The teacher can approach work on radicals by using a quantitative approach. That is, he can focus primarily on radicals as lengths, and then work with radicals becomes geometrical problem solving at first, with only some numeric or algebraic calculation. Use of a tool like Geometer's Sketchpad will be helpful in this regard, although some of what's mentioned below can occur without GSP.

First question for students: Starting with a square of area 1 square unit (see Figure 1), can you make a square of area 2 square units?

[Comment: What does this offer students as far as addressing the issue of properties of radicals?]

Response: This approach offers students the possibility of building up their knowledge of radicals as lengths (of sides of squares) so as to consider what is possible and not possible in combining and operating with these relatively "new" (for 9th graders) mathematical objects.

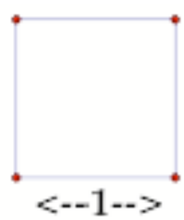


Figure 1

This problem can be solved in multiple ways, see Figure 2 below.

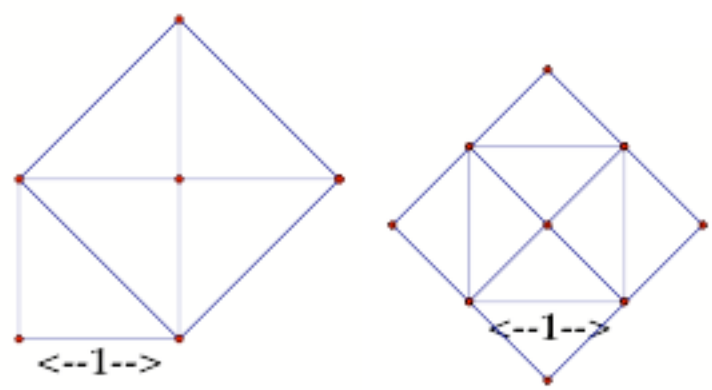


Figure 2

If a square with area 1 square unit has side root 1, which is 1, then a square with area 2 has side root 2. Some investigation here could occur with whether or not root 2 is "like the kind of numbers that we know about," i.e., whole numbers or fractions. (Most students will agree it cannot be a whole number because they can't think of a whole number that, when multiplied by itself, is 2. Fractions are another story but can be approached as noted in Focus 3). [Comment: This appears to be one premise as to why one would go through this approach, i.e. Development of number sense with radicals. What number when squared would give you 2? Are there other numbers one can access in the same manner? Now students can factor the difference of two squares for any $a^2 - b^2$. Is the notion for the difference of two squares that "a & b perfect squares" relevant anymore once students can think about what it means to multiply a number by itself to get a non-perfect square?]

Response: Actually, from my point of view, this approach does open up the notion that any number, or eventually any binomial expression that includes an unknown, can be considered as the difference of two squares. Rather than being a limitation, as the question seems to imply, I believe this can be beneficial in conceiving of, for example, $x^2 - 7$ as the difference of two squares and factorable as

$$(x - \sqrt{7})\ (x + \sqrt{7})$$

Can students make squares of other areas using some of the techniques they have tried so far in making the square of area 2 (e.g., drawing diagonals and using the isosceles right triangles that are formed, circumscribing squares, etc.)? If the circumscribed technique (right side of Figure 2) is continued, squares with areas 4, 8, 16, etc., can be made, and so lengths that are root 4 (a whole number), root 8 (not a whole number), root 16, etc. Note that all of this work can also be confirmed by the Pythagorean Theorem, examining the largest isosceles triangles formed at each iteration.

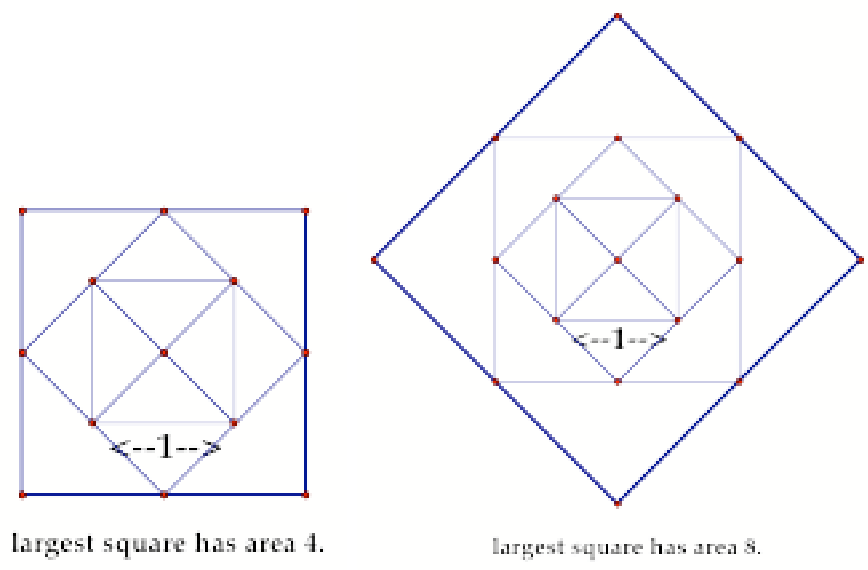


Figure 3

What about a square of area 3? How can this square be made? See Figure 4 below and **page 1** of the GSP sketch VIG C 092905 **radicals.gsp**). [Comment: What properties of radicals are being addressed here?]

Response: In order to address properties of radicals using a quantitative approach, I believe I need to build up how students can conceive of radicals as quantities (lengths of sides of squares).

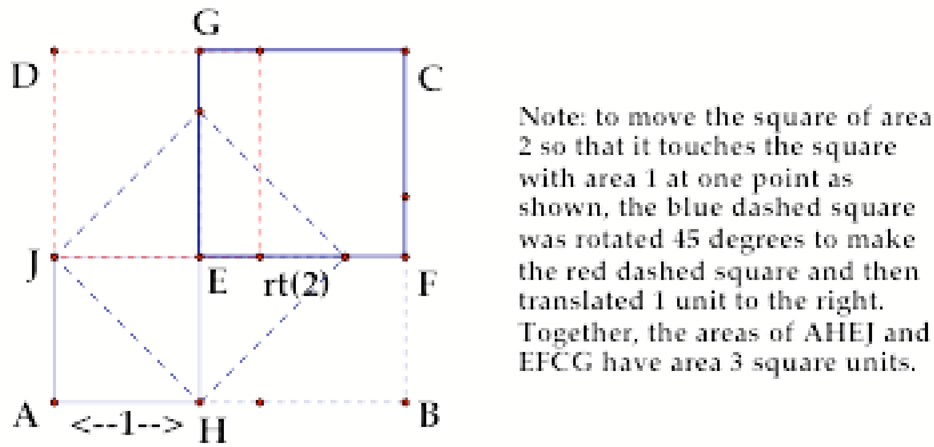


Figure 4

This sketch shows one way to "combine" a square of area 1 (AHEJ) and a square of area 2 (EFCG), and ask students to determine how they can use the diagram to produce a square of area 3. Note that using the sketch to make a square of area 3 is an example of the Pythagorean Theorem. [Comment: How, at this point, can this be determined?]

Response: The area of AHEJ is 1 square unit and the area of EFCG is 2 square units. The sum of these areas is 3 square units. So the area of ABCD is 3 square units plus the areas of rectangles JEGD and HBFE (which are equal). Rearranging via the diagram shown below (figure 5) yields a single square, JLKG, plus the areas of the 2 congruent rectangles (which are now cut into congruent right triangles). So, the area of JLKG has an area of 3 square units and therefore the length of the side of this square is

$$\sqrt{3}$$

The biggest square, ABCD, consists of an area of 1 square unit, an area of 2 square units, and two rectangles that are each 1 unit by root 2 units. When the areas of the two rectangles are cut apart into 4 triangular areas (with legs 1 and root 2 units) and separated, the remaining area in ABCD should also have area 1 + 2 = 3 (see Figure 5).

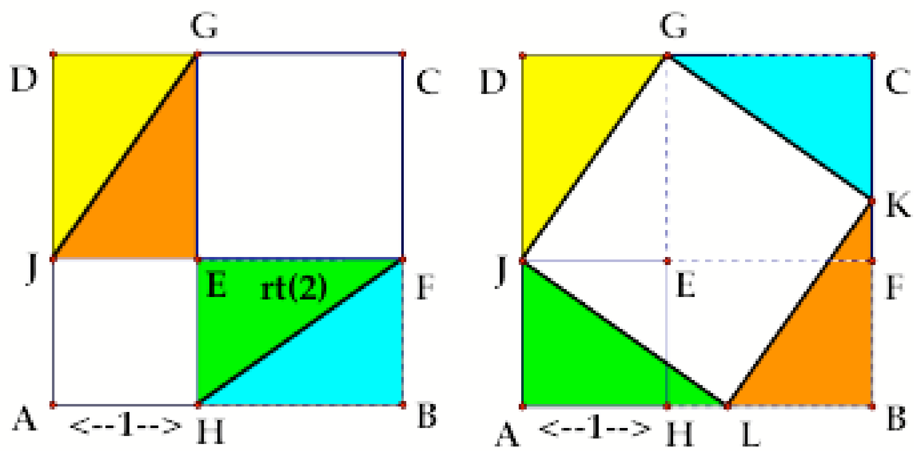


Figure 5

It remains to justify that this remaining area (GKLJ in Figure 5) is a square, which can be done using the 4 congruent right triangles (i.e., GKLJ is at least a rhombus and then it must have at least one right angle because, for example, angles AJL and DJG are complementary). So the sides of square GKLJ can be said to be root 3. Then the teacher can ask students to use what they know to make a square with area 5, then a square with areas 6, 7, and beyond. Note that here the Pythagorean Theorem is used directly as a theorem about areas to make these area relationships and resulting lengths of root 5, root 6, root 7, etc. Students can be challenged to locate all of these lengths on a number line in order to "fill in" the number line and to develop a sense of the relationship of these radicals to numbers like the whole numbers and fractions, which they already know something about (see **pages 2, 3, and 4** of the GSP sketch VIG C 092905 **radicals.gsp**).

Mathematical Focus 2—roots of fractions

The iterative approach to generating squares with areas that are powers of 2, described in Mathematical Focus 1, can go in the "other direction" to generate squares with areas that are fractions (i.e., starting with negative powers of 2). This exploration can lead to students developing radicals of some fractions (negative powers of 2).

The question of how to construct the square root of 1/3 is not trivial! One possibility is to take a square of area 3 and subdivide it horizontally and vertically into thirds, thereby creating 9 squares with area 1/3. One side of these squares must be the square root of 1/3, and is also root 3 divided by 3 (thereby showing that the two must be equal). A similar approach can be taken to generate roots of other unit fractions.

[Comment: Another way to address the students who "simplified the reciprocal of root 3 to root 3 divided by 9" would be through the use of multiplicative inverses. The reciprocal of root 3 is the multiplicative inverse of root 3. Thus, if the reciprocal of root 3 were equivalent to root 3 divided by 9, the product of root 3 divided by 9 and root 3 should equal 1. But, $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{9} = \frac{1}{9}$, not 1.]

Response: I agree, but in this discussion I was trying to develop a quantitative approach, as described previously. One question is whether it is useful to use the vignettes and foci discussion in this way (i.e., to develop several parts of a particular mathematical approach to a situation posed in a vignette), or whether they must have the form of shorter, multiple approaches.

Then, combinations of methods already described can be used to generate roots of non-unit fractions. For example, how can the square root of 2/3 be generated? One way is to use the circumscribing method on the square with area 1/3 to create a square with area 2/3.
[Comment: Is this focus more about deriving the square roots of fractions or is it more? Still seems pedagogical in nature. How is it different than the focus 1?]

Response: This focus is not different from the focus 1. As you have suggested, in this case I have used the foci discussion to develop a particular mathematical approach to addressing the issue of the nature of radicals and how to combine and operate with them.

Mathematical Focus 3—is root 2 a fraction?

Although teachers may know a formal proof to show that this is not the case, the reasoning behind the proof—and ways to bring that reasoning out of students' thinking—is a different kind of mathematical knowledge than just knowing a formal proof (I'll argue).

One way to approach this question with students is to assume that root 2 is a fraction, which can be written in lowest terms as a/b (a, b are non-zero whole numbers). If this is the case, then 2 = a2/b2, and 2b2 = a2. So this implies that a2 is even and so a is even (students will have to justify why!). But that means that there will be two factors of 2 in a2, so there must be two factors of 2 in 2b2. That means that b2 has to have one factor of 2 (can it?). Since if b2 has a factor of 2, it must have two factors of 2, and so b must have a factor of 2, then a and b both are even—but we assumed that the fraction a/b was written in lowest terms! So root 2 must not be able to be written as a fraction we know about, a/b in lowest terms where a and b are both non-zero whole numbers. [Comment: This focus doesn't seem relative to the question of the vignette. Seems disconnected in nature. What does the fact that rt2 can't be rational? Are you saying that establishing the classification of numbers: natural, rational, irrational, etc. is important for understanding properties of radicals?]

Response: Yes, I believe that establishing the notion of this mathematical object, a radical, in relation to other more familiar mathematical objects like natural numbers and fractions, is crucial to constructing radicals as numbers, let alone combining and operating with them. So this focus continues the development of this quantitative approach toward that end.

Mathematical Focus 4—addition and multiplication of radicals

One of the big issues in the vignette is why intuitions about addition, based on addition of whole numbers, do *not* hold with fractions or radicals, while intuitions about multiplication (again based on multiplication of whole numbers) *do*. [Comment: The vignette statement seems to address only radicals and not rational numbers.] So a major reason for taking a quantitative approach to work with radicals as outlined above is to investigate this issue, as I briefly describe here.

Response: Please see my previous response. Part of the reason, I believe, that students will add

$$\sqrt{2}+\sqrt{3}$$

and get

$$\sqrt{5}$$

has to do with their familiarity with whole numbers as mathematical objects. So I believe that a major issue to address in this situation is why those intuitions (built up over years!) do not hold. Note that students hardly have the same amount of time to build up intuitions about combining and operating with radicals.

After radicals have been constructed as the lengths of sides of squares, questions can arise about how to combine these lengths. Lengths can certainly be added by joining them contiguously, but, since root 2 and root 3 cannot be written as whole numbers or fractions, we have no way to know whether we can notate their combination with a single graphic item, the way we can combine 2.5 and 3.75 into 6.25. So (at least for the moment), root 2 + root 3 is exactly that, root 2 + root 3. Furthermore, using lengths it is possible to develop intuitions about root 2 and root 3 NOT being equal to root 5 (see **page 5** of VIG C 092905 **radicals.gsp**). However, we can combine multiple lengths of root 2 by determining how many root 2's we have (e.g., 5 root 2).

[Comment: Another approach would be to provide a counterexample, such as

$$\sqrt{16}+\sqrt{25}=4+5=9$$
$$\sqrt{16}+\sqrt{25}\neq\sqrt{41} \text{ since } \sqrt{41}\neq 9$$

.]

Response: I agree that a counterexample can be useful, but even several numerical counterexamples do not allow students to construct radicals as numbers and some kind of rationale for why intuitions about addition from whole numbers do not apply.

Multiplication of two radicals like root 2 and root 3 can be thought about using similar triangles (see **page 6** of VIG C 092905 **radicals.gsp**). Work with commutativity and associativity can occur by thinking about finding volumes of rectangular prisms that have radicals as lengths of sides (see **page 7** of VIG C 092905 **radicals.gsp**).

[Comment: This is a listing of properties that one can apply to radicals with some of your reasoning built in to clarify. Where is it going?]
Response: The point of this final focus is to think about ways to combine and operate with radicals.

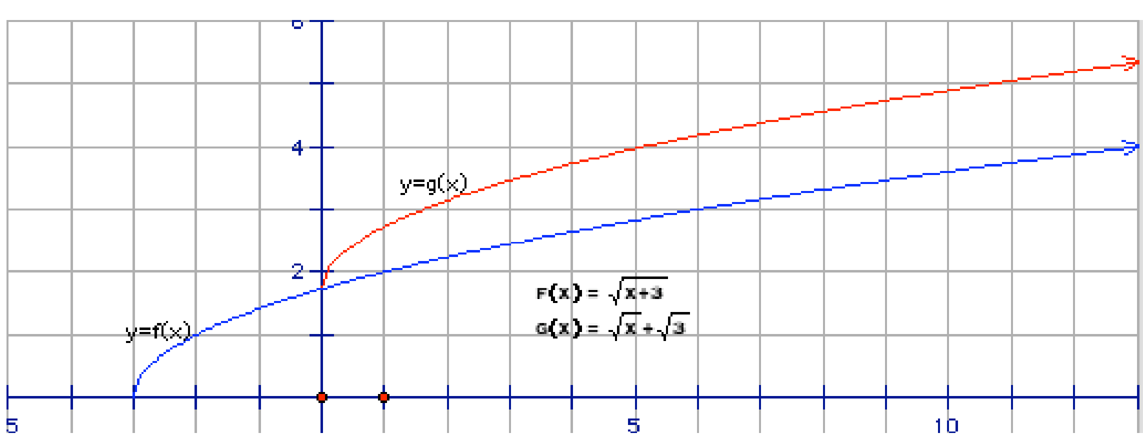
Possible Focus . . .
The figure above is another way one can create the lengths of segments with radical forms and relate them in an iterative process. In addition, it is possible to reason with students about the length of rt2 being less than 2, but longer than 1. How can we use the figure to describe the relationships among other square roots?

Response: This seems like a reasonable addition, although I don't know which figure is being referred to.

Another focus . . .
What number do you square to get the value of a non-perfect square? What is the relationship between these numbers? What is the criteria one must use to locate such a number?
When solving a quadratic of the form $a^2 - b^2 = 0$ and $a = 3x^2$ and $b = 5$ . . . How can one use factoring and the principle of zero products to solve this equation? What are we really asking here?

Response: It seems like this focus goes beyond the situation?

Another thought...
If students think that rt2+rt3 = rt5 then they are saying that rt2+rt3 = rt(2+3). In addition, they are developing a misconception (misrepresented property) of radicals. Is there ever a possibility when rt A + rt B=rt(A+B)? What are the conditions for such a generalization to hold true? What if students explored the equation as a system of equations rtx + rt3 = y and y=rt(x+3) maybe students can see a difference in there graphs to determine the validity of their mis-conception. See the figure below:



Response: My general comment is that a system of equations such as this may presume a great deal of knowledge that students have not yet built up. However, I do not disagree that a teacher might want to know this.

Return to the Sit32 Page

[1] The following discussion is adapted from Leslie P. Steffe's course, EMAT 7080.