Situation 35: Solving Quadratic Equations
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Prompt

In an Algebra 1 class some students began solving a quadratic equation as follows:

Solve for *x*:

$$x^2 = x + 6$$
$$\sqrt{x^2} = \sqrt{x+6}$$
$$x = \sqrt{x+6}$$

They stopped at this point, not knowing what to do next.

Commentary

Quadratic equations are an important part of the curriculum of school mathematics, and solving them is a necessary skill. These equations can be solved in a number of ways, and teachers ought to have each method mastered in order to assist their students in correctly solving quadratic equations.

The point at which these students stopped in the problem above is not a solution because *x* has not been isolated. Teachers must have a clear understanding of what constitutes a solution. The students' work above can be used to solve the quadratic (as will be seen in the Mathematical Foci which follow). However, $x = \sqrt{x+6}$ is not a solution in itself because it does not show what value of *x* makes the equation true.

Focus 1 addresses a graphical approach to the problem above: the final equation $x=\sqrt{x+6}$ is equivalent $x^2=x+6$ if and only if its graph yields the same solution(s) as $x^2=x+6$.

Foci 2 and 3 present two accurate methods of solving a quadratic equation: factoring, and the quadratic formula. These are included because this Prompt illustrates the importance of having accurate and certain means by which to solve quadratic equations.

Focus 4 offers a graphical method of solving the quadratic equation. However, instead of only getting a partial solution (as the student in this Prompt did), Focus 4 gives both solutions of the equation. On the way to these two solutions, one of the definitions of absolute value is used: $|x|=\sqrt{x^2}$.

Focus 5 gives a method of solving the equation numerically by using a table of values. Though the solutions for this Situation are integers, solutions of quadratics are not always integer values. For this reason, creating a table is not a preferable method for all quadratics. It is, however, a useful tool in this case.

Mathematical Foci

Mathematical Focus 1

The solutions to the three equations

$$x^2=x+6$$
$$\sqrt{x^2}=\sqrt{x+6}$$
$$x=\sqrt{x+6}$$

can be compared graphically to determine whether the equations are equivalent. Equations are equivalent if and only if they have the same solutions.

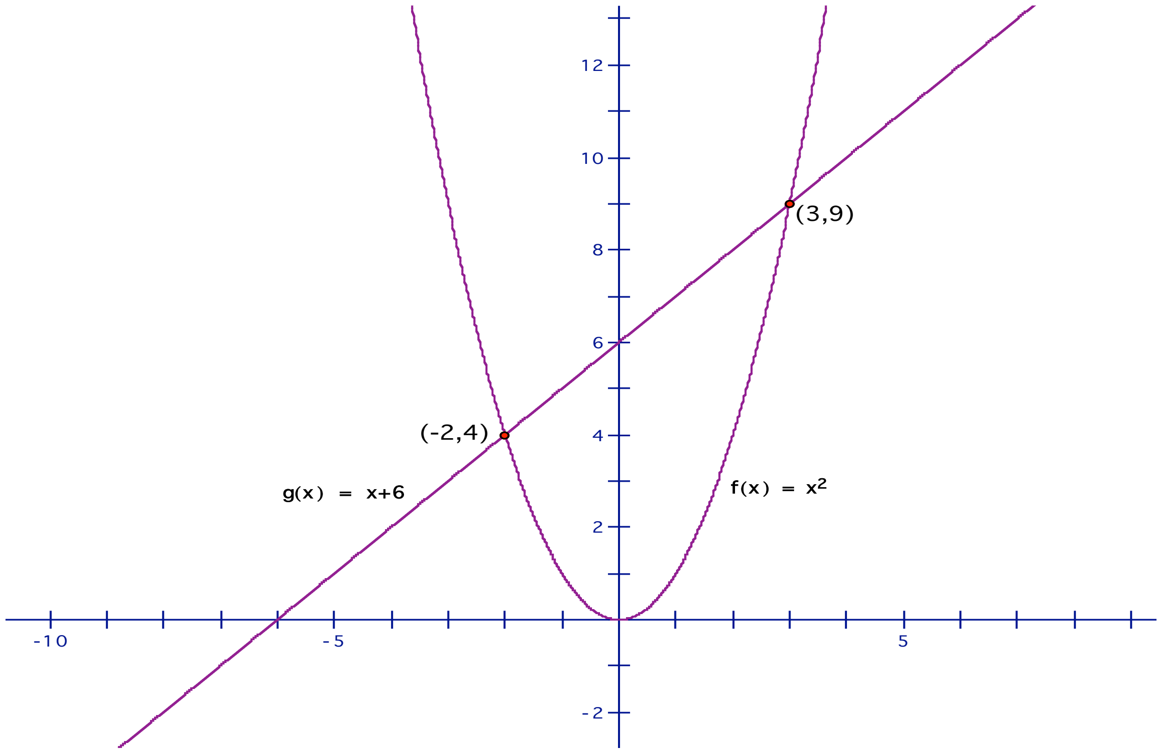

Figure 1: $f(x) = x^2$ and $g(x) = x + 6$ intersect at x = -2, 3.

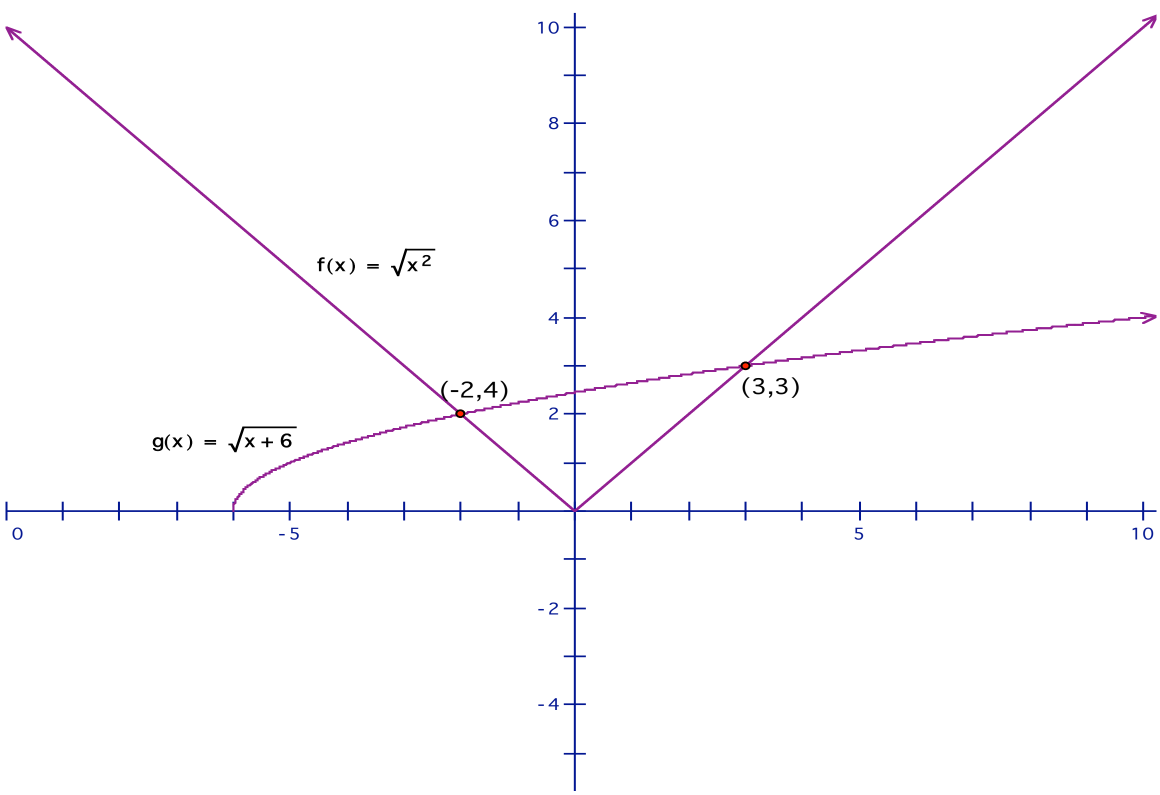

Figure 2: $f(x) = \sqrt{x^2}$ and $g(x) = \sqrt{x+6}$ intersect at x = -2, 3.

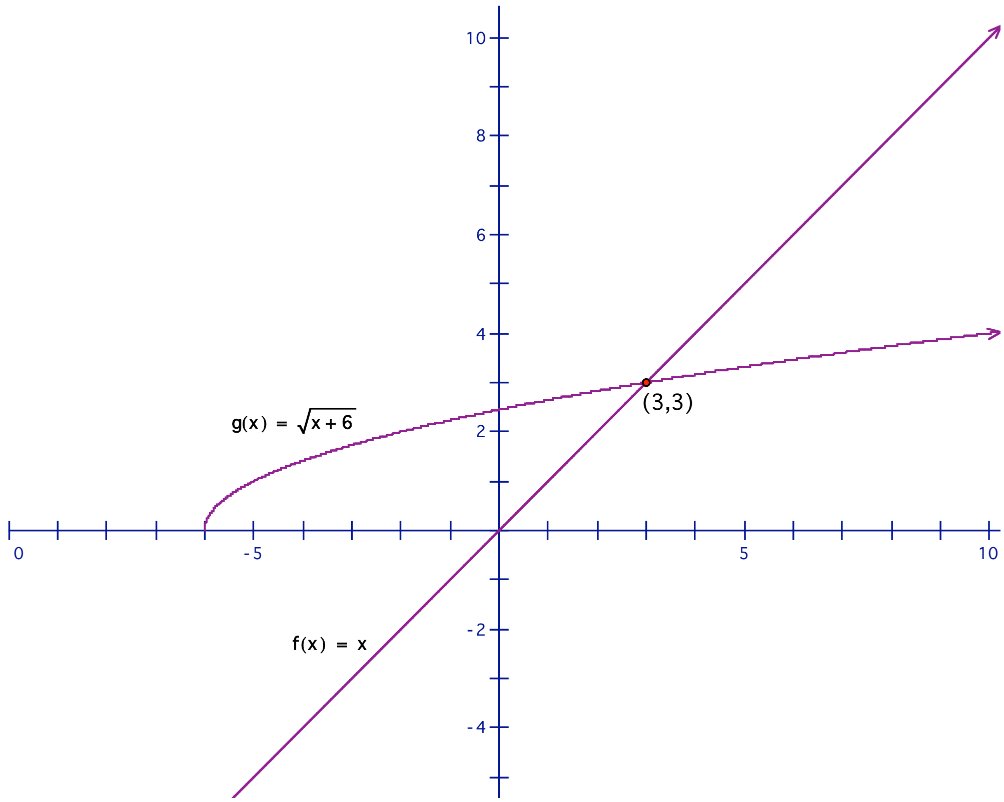

Figure 3: $f(x) = x$ and $g(x) = \sqrt{x+6}$ intersect at *x* = 3.

The last equation, $x = \sqrt{x+6}$, is not equivalent to the other two equations since its solution is not the same as that of the other equations. Therefore $x = \sqrt{x+6}$ is not equivalent to $x^2 = x + 6$. However, since the graphs in Figure 1 and 2 have the same solutions, those equations ($x^2 = x + 6$ and $\sqrt{x^2} = \sqrt{x+6}$) <u>are</u> equivalent.

Mathematical Focus 2

The quadratic formula can be used to solve $x^2 = x + 6$:

$$
\begin{aligned}
&x^2 = x + 6 \\
&x^2 - x - 6 = 0 \\
&x = \frac{1 \pm \sqrt{1 - 4(1)(-6)}}{2} \\
&= \frac{1 \pm 5}{2} \\
&= 3, -2
\end{aligned}
$$

Mathematical Focus 3

The quadratic equation, $x^2 = x + 6$, can be solved by factoring and applying the zero product property:

$$
\begin{aligned}
&x^2 = x + 6 \\
&x^2 - x - 6 = 0 \\
&(x - 3)(x + 2) = 0 \\
&x = 3, -2
\end{aligned}
$$

Mathematical Focus 4

The equation, $x^2 = x + 6$, can be solved graphically. In taking the steps shown below, one arrives at an equation involving $\sqrt{x^2}$. It is worth noting here that one of the definitions of absolute value is $|x| = \sqrt{x^2}$. When solving absolute value equations, more than one solution is possible. For example, in the equation $|x|$ = 3, the solutions for *x* are 3 and -3. Similarly, when working to solve the equation $\sqrt{x^2} = \sqrt{x+6}$, *x* is equivalent to $\sqrt{x+6}$ and $-\sqrt{x+6}$. In order to include this information, absolute value notation has been introduced in the following series of equations.

$$x^2 = x + 6$$

$$\sqrt{x^2} = \sqrt{x+6}$$

$$|x| = \sqrt{x+6}$$

$$x = \pm\sqrt{x+6}$$

In order to proceed with the graphical solution, graph $f(x) = x$, $g(x) = \sqrt{x+6}$, and $h(x) = -\sqrt{x+6}$. The point(s) at which *f(x)* intersect(s) the other curves will indicate the solution(s).

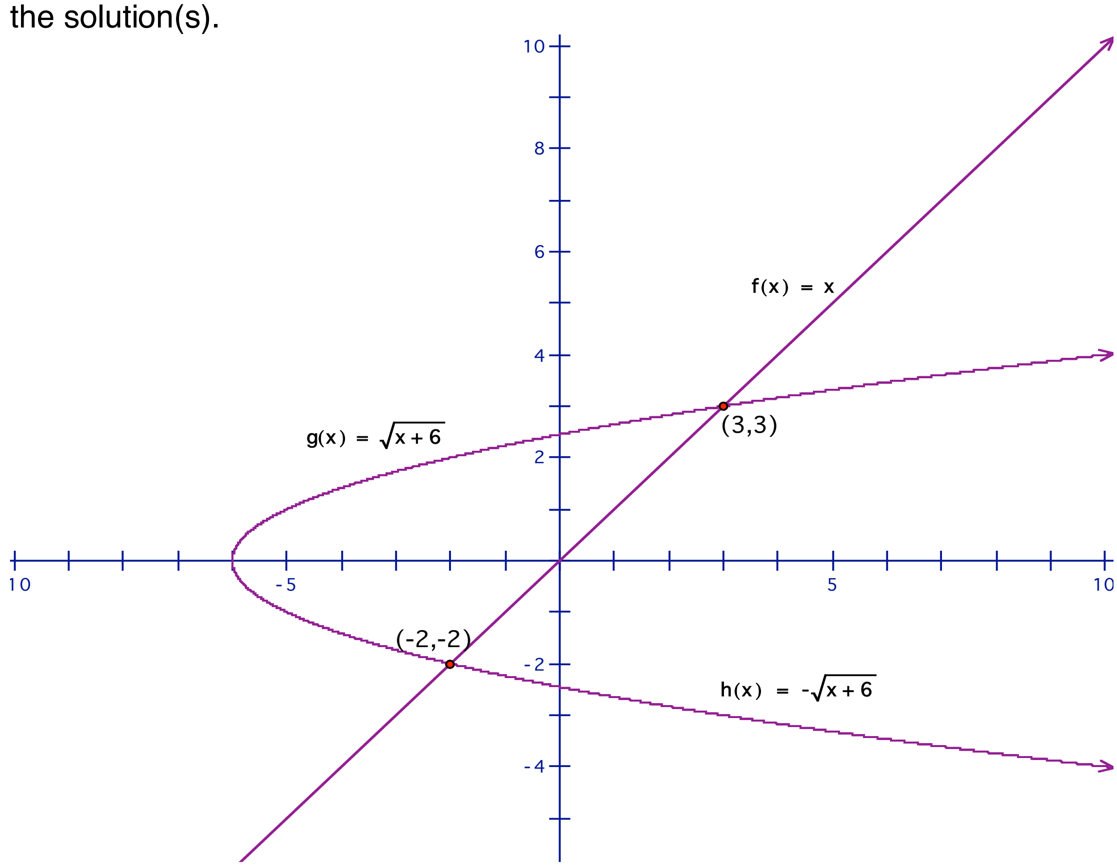

Figure 4: $f(x) = x$, $g(x) = \sqrt{x+6}$, and $h(x) = -\sqrt{x+6}$ intersect at *x* = -2, 3.

Mathematical Focus 5

A table is a useful tool in determining the behavior of functions for certain discrete values of x. An appropriate table for this Situation would include values for x and the resulting values for x^2 and $x + 6$.

x	x^2	$x + 6$
-4	16	2
-3	9	3
-2	**4**	**4**
-1	1	5
0	0	6
1	1	7
2	4	8
3	**9**	**9**
4	16	10
5	25	11

The x-values which produce equivalent results for x^2 and $x + 6$ are solutions to the equation $x^2 = x + 6$. These values, as seen above, are $x = -2$ and $x = 3$.