Situation 35: Solving Quadratic Equations
Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
June 30, 2005 – Jeanne Shimizu

University of Georgia
September 14, 2006 – Sarah Donaldson

<u>Prompt</u>

In an Algebra 1 class some students began solving a quadratic equation as follows:

Solve for *x*:

$$x^2 = x + 6$$
$$\sqrt{x^2} = \sqrt{x + 6}$$
$$x = \sqrt{x + 6}$$

They stopped at this point, not knowing what to do next.

<u>Commentary</u>

This Situation provides an opportunity to highlight some issues concerning solving equations (both in general and specifically regarding quadratic equations) that are prevalent in school mathematics.

Focus 1 provides guidelines for solving any algebraic equation and emphasizes maintaining equivalence. Focus 2 shows the relationship between the solution(s) of an equation and the zero(s) of a function. This Focus contains a graphical approach to solving quadratic equations. Foci 3 and 4 present two accurate methods of solving a quadratic equation: factoring, and the quadratic formula. These are included because this Prompt illustrates the importance of having accurate and certain means by which to solve quadratic equations. Focus 5 provides a geometric approach for solving $x^2 = x + 6$.

Mathematical Foci

Mathematical Focus 1: Solving equations

In order to solve an algebraic equation, one must determine the value(s) for the unknown(s) that satisfy the equation (i.e. make the equation true). In this Situation, *x* is an unknown quantity. Simply getting an equation which begins "*x* =" (such as $x = \sqrt{x+6}$) does not constitute a solution. The *x* must be expressed in terms of that which does not involve *x* (that is, *x* must be isolated).

A common strategy for solving equations is algebraic manipulation: performing operations in order to isolate the unknown quantity. Solving equations in this way requires following certain rules. At each step, equivalence must be maintained from one equation to the next. This is done in two ways:

a) by keeping the equation balanced (ex. by adding 3 to both sides), and
b) by insuring that each equation in the process yields the same solution(s) as the original equation.

For example:

$$2x + 7 = 15$$
$$-7 \quad -7$$
$$2x = 8$$
$$(2x)/2 = 8/2$$
$$x = 4$$

Balance is kept by performing the same operations (subtracting 7, dividing by 2) on both sides. Also, each step (in this case, $2x = 8$ and $x = 4$) yields the same solution as $2x + 7 = 15$.

The example in this Situation is $x^2 = x + 6$. Though balance is kept by taking the square root of both sides of the equation, equivalence is not maintained in the following step:

$$\sqrt{x^2} = \sqrt{x+6}$$
$$x = \sqrt{x+6}$$

because $x = \sqrt{x+6}$ has only one solution ($x = 3$), while $x^2 = x + 6$ has two. This can be seen below, in which equivalence is maintained.

$$x^2 = x + 6$$
$$\sqrt{x^2} = \sqrt{x+6}$$
$$|x| = \sqrt{x+6}$$
$$x = \pm\sqrt{x+6}$$

In taking these steps, one arrives at an equation involving $\sqrt{x^2}$. It is worth noting here that one of the definitions of absolute value is $|x| = \sqrt{x^2}$. When solving absolute value equations, more than one solution is possible. For example, in the equation $|x| = 3$, the solutions for *x* are 3 and -3. Similarly, when working to solve the equation $\sqrt{x^2} = \sqrt{x+6}$, *x* is equivalent to $\sqrt{x+6}$ and $-\sqrt{x+6}$. However, as noted earlier, $x = \pm\sqrt{x+6}$ does not provide a solution to the original equation.

This discussion may communicate that taking the square root of both sides of an equation is never a good idea. This is not the case. There are many instances in which taking a root of both sides is a good step toward arriving at a solution. In fact, this equation, if it had not been for the *x* term, could have been solved that way.

Mathematical Focus 2: Equations and functions

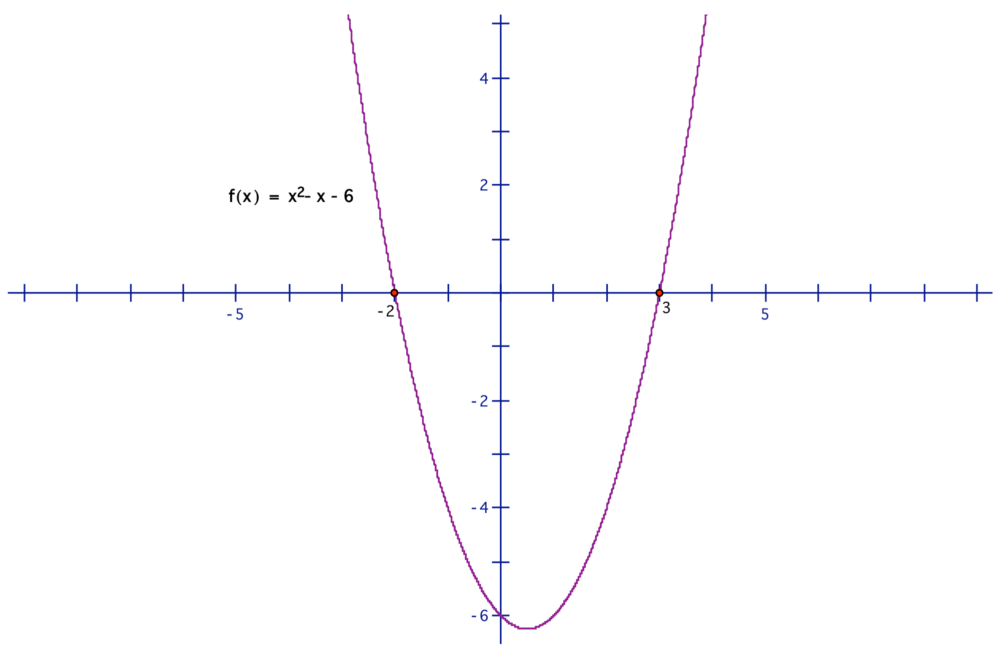
The solutions to an equation in which an expression involving x is equal to zero (such as $x^2 - 4 = 0$) are comparable to the zeros of a function of x (such as $f(x) = x^2 - 4$). This is because a zero, or x-intercept, of a function is the x-value for which the value of $f(x)$ is zero.

For example, the solutions to $x^2 - 4 = 0$ are $x = 2$ and $x = -2$; and the zeros (x-intercepts) of $f(x) = x^2 - 4$ are $x = 2$ and $x = -2$. The equation $x^2 - 4 = 0$ and the function $f(x) = x^2 - 4$ are not the same, as x is an unknown in the equation (represents a specific value) while in the function, x is a variable (changing quantity). However, the equation and the function are related: the solutions of the equation are the same as the zeros of the function.

Since solutions to equations and zeros of functions are related in this way, graphing a function can be a useful method of solving an equation. However, as noted above, if the strategy is to find the zeros of the function, the accompanying equation must be equal to zero. In this Situation, this will involve manipulating the equation and setting it equal to zero:

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$

The graph of $f(x) = x^2 - x - 6$ will indicate, by its zeros, the solutions of $x^2 - x - 6 = 0$.



A similar method requires graphing the functions $f(x) = x^2$ and $g(x) = x + 6$ (that is, treat each side of the original equation as a function) and determine their points of intersection. These are the points at which x^2 and $x+6$ are equal. We forego this method here, as it is better employed in other Situations.

Mathematical Focus 3: Factoring

The quadratic equation $x^2 = x + 6$ can be solved by factoring and applying the Zero-Product Property. The Zero-Product Property states:

> If $a \cdot b = 0$, then $a = 0$, $b = 0$, or $a = b = 0$.

This Property is critical in this Situation because it is the key rule that allows the equation to be solved by factoring (as will be seen shortly). It is worth noting that this Property is unique to zero. That is, there is no "Two-" or "Six-Product Property." A common mistake is to generalize the Property to something like:

> If $a \cdot b = \text{c}$, then $a = \text{c}$ or $b = \text{c}$.

This error might be used, for example, in the following way:

$$x^2 + 7x + 12 = 2$$
$$(x + 3)(x + 4) = 2$$
$$x + 3 = 2 \qquad x + 4 = 2$$
$$x = -1 \qquad x = -2$$

However, zero is the only number for which the Property holds. It is correctly used by first setting $x^2 = x + 6$ equal to zero then factoring and setting each factor equal to zero:

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x - 3 = 0$$
$$x + 2 = 0$$
$$x = 3, -2$$

Mathematical Focus 4: Quadratic Formula

Solutions of quadratic equations are not always integers. Nor are they necessarily real numbers. For these and other reasons, the quadratic formula is a useful tool with which to solve a quadratic equation.

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$x = \frac{1 \pm \sqrt{1 - 4(1)(-6)}}{2}$$
$$= \frac{1 \pm 5}{2}$$
$$= 3, -2$$

Mathematical Focus 5: Geometric approach

One way to approach the equation $x^2 = x + 6$ geometrically is to think of it in terms of areas: *x* is an unknown value for which the area of a square with side length *x* is the same as the area of a rectangle with area *x* + 6. See GSP sketch here.

This sketch reveals that the *x*-values for which the two areas are the same (and therefore the *x*-values that are solutions to $x^2 = x + 6$) seem to be 3 and -2.