MAC-CPTM Situations Project

Situation 35: Solving Quadratic Equations

Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
June 30, 2005 – Jeanne Shimizu

Revised at
University of Georgia
Center for Proficiency in Teaching Mathematics
September 14, 2006 – Sarah Donaldson
November 12, 2008 – Kelly Edenfield and Erik Jacobson

Prompt

In an Algebra 1 class some students began solving a quadratic equation as follows:

Solve for *x*:

$$x^2 = x + 6$$
$$\sqrt{x^2} = \sqrt{x+6}$$
$$x = \sqrt{x+6}$$

They stopped at this point, not knowing what to do next.

Commentary

This Situation provides an opportunity to highlight some issues concerning solving equations (both specifically regarding quadratic equations and in general) that are prevalent in school mathematics.

Foci 1 and 2 present two accurate methods of solving a quadratic equation: factoring and the quadratic formula. These are included because this Prompt illustrates the importance of having accurate and certain means by which to solve quadratic equations. Focus 3 provides a geometric approach for solving $x^2 = x + 6$. Focus 4 provides guidelines for solving any algebraic equation and emphasizes maintaining equivalence. Focus 5 shows the relationship between the solution(s) of an equation and the zero(s) of a function. This Focus contains a graphical approach to solving quadratic equations. The Situation ends with a Post-Commentary on the occurrence of extraneous solutions.

Mathematical Foci

Mathematical Focus 1

Factoring and using the Zero Product Property can be used to solve many quadratic equations.

The quadratic equation $x^2 = x + 6$ can be solved by factoring and applying the Zero-Product Property. The Zero-Product Property states:

If $a \cdot b = 0$, then $a = 0$, $b = 0$, or $a = b = 0$.

The Zero-Product Property is a consequence of the real number system. The real numbers are an algebraic *field* and therefore also an *integral domain* since every field is also an integral domain. (Note, however, that the converse is false; the integers are an integral domain but not a field because not every integer has a multiplicative inverse.) A defining property of integral domains is that there are no *zero-divisors*. Zero-divisors are non-zero elements whose product is zero.

For example, consider the ring Z[6], that is, the set {0,1,2,3,4,5} with addition and multiplication defined modulus 6. The element 2 and the element 3 are non-zero but their product is the element 0 (mod 6). Thus Z[6] is not an integral domain and a quadratic equation in Z[6] could not be solved using the Zero-Product Property.

Since the real numbers are an integral domain, the Zero-Product Property can be used to solve equations of real numbers. In fact, this Property is critical in this Situation because it is the key rule that allows the equation to be solved by factoring (as will be seen shortly). It is worth noting that this Property is unique to zero. That is, there is no "Two-" or "Six-Product Property." A common mistake is to generalize the Property to something like:

If $a \cdot b = c$, then $a = c$ or $b = c$.

This error might be used, for example, in the following way:

$$x^2 + 7x + 12 = 2$$
$$(x + 3)(x + 4) = 2$$
$$x + 3 = 2 \qquad x + 4 = 2$$
$$x = -1 \qquad x = -2$$

Zero is the only number for which the Property holds because zero is the only number that has only itself for factors. (See the Zero-Product Property Situation for additional information.) In this example, the Property is correctly used by first setting $x^2 = x + 6$ equal to zero, factoring, and then setting each factor equal to zero:

$$
\begin{aligned}
&x^2 = x + 6 \\
&x^2 - x - 6 = 0 \\
&(x-3)(x+2) = 0 \\
&x - 3 = 0 \\
&x + 2 = 0 \\
&x = 3, -2
\end{aligned}
$$

Mathematical Focus 2

All quadratic equations can be solved by completing the square or by employing the use of the quadratic formula.

Solutions of quadratic equations are not always integers, nor are they necessarily real numbers. For these and other reasons, the quadratic formula, derived by completing the square on $ax^2 + bx + c = 0$, is a useful tool with which to solve a quadratic equation.

$$
\begin{aligned}
&x^2 = x + 6 \\
&x^2 - x - 6 = 0 \\
&x = \frac{1 \pm \sqrt{1 - 4(1)(-6)}}{2} \\
&x = \frac{1 \pm 5}{2} \\
&x = 3, -2
\end{aligned}
$$

Mathematical Focus 3

A geometric approach that uses area models can be used to represent quadratic equations and their solutions.

One way to approach the equation $x^2 = x + 6$ geometrically is to think of it in terms of areas: *x* is an unknown value for which the area of a square with side length *x* is the same as the area of a rectangle with area *x* + 6. See GSP sketch here.

This sketch reveals that the *x*-values for which the two areas are the same (and therefore the *x*-values that are solutions to $x^2 = x + 6$) seem to be 3 and -2.

Mathematical Focus 4

Solving an equation through algebraic manipulation requires that equivalence is maintained between each form of the equation.

In order to solve an algebraic equation, one must determine the value(s) for the unknown(s) that satisfy the equation (i.e. make the equation true). In this Situation, *x* is an unknown quantity. Simply getting an equation which begins "$x =$" (such as $x = \sqrt{x+6}$) does not constitute a solution. The *x* must be expressed in terms of that which does not involve *x* (that is, *x* must be isolated).

A common strategy for solving equations is algebraic manipulation: performing operations in order to isolate the unknown quantity. Solving equations in this way requires following certain rules. At each step, equivalence must be maintained from one equation to the next. This is done in two ways:

a) by keeping the equation balanced (ex. by adding 3 to <u>both sides</u>), and
b) by insuring that each equation in the process yields the same solution(s) as the original equation.

For example:

$$\begin{aligned} 2x + 7 &= 15 \\ -7 \quad & \;\; -7 \\ 2x &= 8 \\ \frac{2x}{2} &= \frac{8}{2} \\ x &= 4 \end{aligned}$$

Balance is kept by performing the same operations (subtracting 7, dividing by 2) on both sides. Also, each step (in this case, $2x = 8$ and $x = 4$) yields the same solution as $2x + 7 = 15$.

The example in this Situation is $x^2 = x + 6$. Though balance is kept by taking the square root of both sides of the equation, equivalence is not maintained in the following step:

$$\sqrt{x^2} = \sqrt{x+6}$$
$$x = \sqrt{x+6}$$

because $x = \sqrt{x+6}$ has only one solution, while $x^2 = x + 6$ has two.
(See the Post-Commentary for an explanation of why $x = \sqrt{x+6}$ has only one solution.)

Below, however, equivalence <u>is</u> maintained.

$$x^2 = x + 6$$
$$\sqrt{x^2} = \sqrt{x+6}$$
$$|x| = \sqrt{x+6}$$
$$x = \pm\sqrt{x+6}$$

In taking these steps, one arrives at an equation involving $\sqrt{x^2}$. It is worth noting here that one of the definitions of absolute value is $|x| = \sqrt{x^2}$. When solving absolute value equations, more than one solution is possible. For example, in the equation $|x| = 3$, the solutions for x are 3 and -3. Similarly, when working to solve the equation $\sqrt{x^2} = \sqrt{x+6}$, x is equivalent to $\sqrt{x+6}$ and $-\sqrt{x+6}$. However, as noted earlier, $x = \pm\sqrt{x+6}$ does not provide a solution to the original equation.

This discussion may communicate that taking the square root of both sides of an equation is never a good idea. This is not the case. There are many instances in which taking a root of both sides is a good step toward arriving at a solution. In fact, this equation, if it had not been for the x term, could have been solved that way.

Mathematical Focus 5

Equations can be solved by graphically determining the zeros of the associated function.

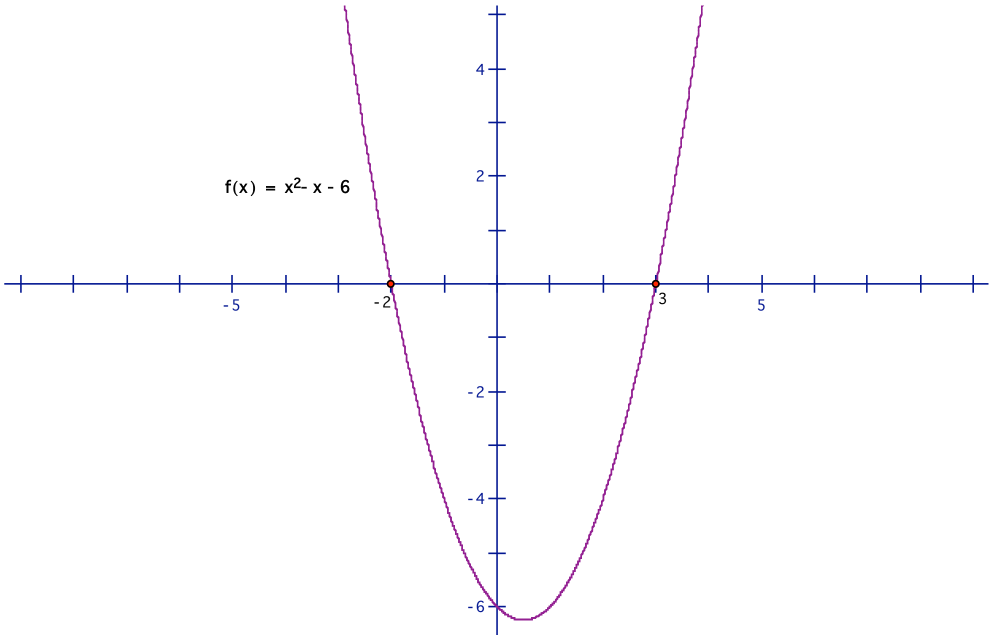
The solutions to an equation in which an expression involving x is equal to zero (such as $x^2 - 4 = 0$) are comparable to the zeros of a function of x (such as $f(x) = x^2 - 4$). This is because a zero, or x-intercept, of a function is the x-value for which the value of $f(x)$ is zero.

For example, the solutions to $x^2 - 4 = 0$ are $x = 2$ and $x = -2$; and the zeros (the x-intercepts) of $f(x) = x^2 - 4$ are $x = 2$ and $x = -2$. The equation $x^2 - 4 = 0$ and the function $f(x) = x^2 - 4$ are not the same, as x is an unknown in the equation (represents a specific value) while in the function, x is a variable (changing quantity). However, the equation and the function are related: the solutions of the equation are the same as the zeros of the function.

Since solutions to equations and zeros of functions are related in this way, graphing a function can be a useful method of solving an equation. However, as noted above, if the strategy is to find the zeros of the function, the accompanying equation must be equal to zero. In this Situation, this will involve manipulating the equation and setting it equal to zero:

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$

The graph of $f(x) = x^2 - x - 6$ will indicate, by its zeros, the solutions of $x^2 - x - 6 = 0$.



A similar method requires graphing the functions $f(x) = x^2$ and $g(x) = x + 6$ (that is, treat each side of the original equation as a function) and determine their points of intersection. These are the points at which x^2 and $x + 6$ are equal. We forego this method here, as it is better employed in other Situations.

Post Commentary

Often, in solving equations, we find extraneous solutions. These result when the original domain is expanded during the course of the solution process.

It was stated earlier that $x^2 = x + 6$ and $x = \sqrt{x+6}$ could not be equal because they have different solution sets, specifically that the former has two solutions and the latter has only one.

To see that $x = \sqrt{x+6}$ has only one solution, consider how to solve the equation, using the principles in Focus 4.

First, however, attention must be given to the domain of $x = \sqrt{x+6}$. For $\sqrt{x+6}$ to be defined in the real numbers, $x \geq -6$. But if $x = \sqrt{\text{expression}}$, $x \geq 0$. This means that the domain is the intersection of $x \geq -6$ and $x \geq 0$, thus the domain is $x \geq 0$.

Now, to solve $x = \sqrt{x+6}$: The inverse operation for taking the square root is squaring. Squaring both sides yields the original problem ($x^2 = x + 6$) and the two solutions are found to be $x = 3, -2$. Note that in this original form of the equation, the domain is the set of real numbers; there are no restrictions on the values of *x*. However, in $x = \sqrt{x+6}$,

$x \geq 0$, so the only solution is $x = 3$. Negative 2 is called an ***extraneous solution***. It was introduced by expanding the valid domain from the non-negative real numbers to all real numbers.

Note that different problems will have extraneous solutions from the expanding of other domains to the real numbers. Consider $\log_3(5x - 12) + \log_3 x = 2$. From the first term, $x > \frac{12}{5}$. From the second, $x > 0$. The intersection of these two restrictions is $x > \frac{12}{5}$. Solving this problem yields two solutions because the problem is converted to a quadratic function with a domain of all real numbers. This domain expansion introduces a possible extraneous solution.

$$
\begin{aligned}
&\log_3(5x - 12) + \log_3 x = 2 \\
&\log_3(5x - 12)(x) = 2 \\
&5x^2 - 12x = 3^2 \\
&5x^2 - 12x = 9 \\
&5x^2 - 12x - 9 = 0 \\
&(5x + 3)(x - 3) = 0 \\
&x = -\frac{3}{5} \quad \text{or } x = 3
\end{aligned}
$$

In fact, the first solution is not in the original domain and is an extraneous solution. The only solution for this logarithmic equation is $x = 3$.