Situation 08: Ladder

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Prompt

A high school geometry class was in the middle of a series of lessons on loci. The teacher chose to discuss one of the homework problems from the previous day's assignment.

A student read the problem from the textbook (Brown, Jurgensen, & Jurgensen, 2000): *A ladder leans against a house. As A moves up or down on the wall, B moves along the ground. What path is followed by midpoint M? (Hint: Experiment with a meter stick, a wall, and the floor.)*

The teacher and two students conducted the experiment in front of the class, starting with a vertical "wall" and a horizontal "floor" and then marking several locations of M as the students moved the meter stick. The teacher connected the points. Their work produced the following data picture on the board:
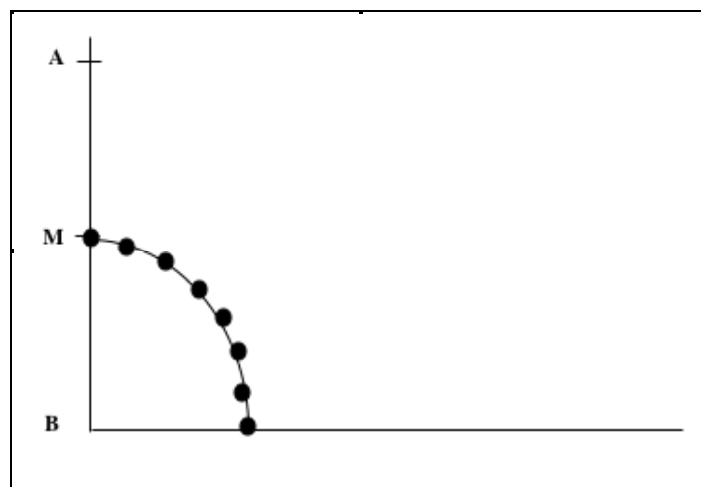
A student commented, "That's a heck of an arc." Is it really an arc?

Commentary

The foci for this situation show a variety of models and representations of the locus. In the class demonstration, the data was generated by a physical model.

Mathematical Foci

Mathematical Focus 1

We create a sample of data points using the Geometer's Sketchpad (VIG M 050606 Ladder arc.gsp based on GSP provided file, Falling Ladder.gsp). Using the marks for midpoint of the ladder when it is vertical and horizontal and the midpoint when it is in one other location, we create an arc that passes through all of the other points. This suggests the path of the midpoint is an arc.
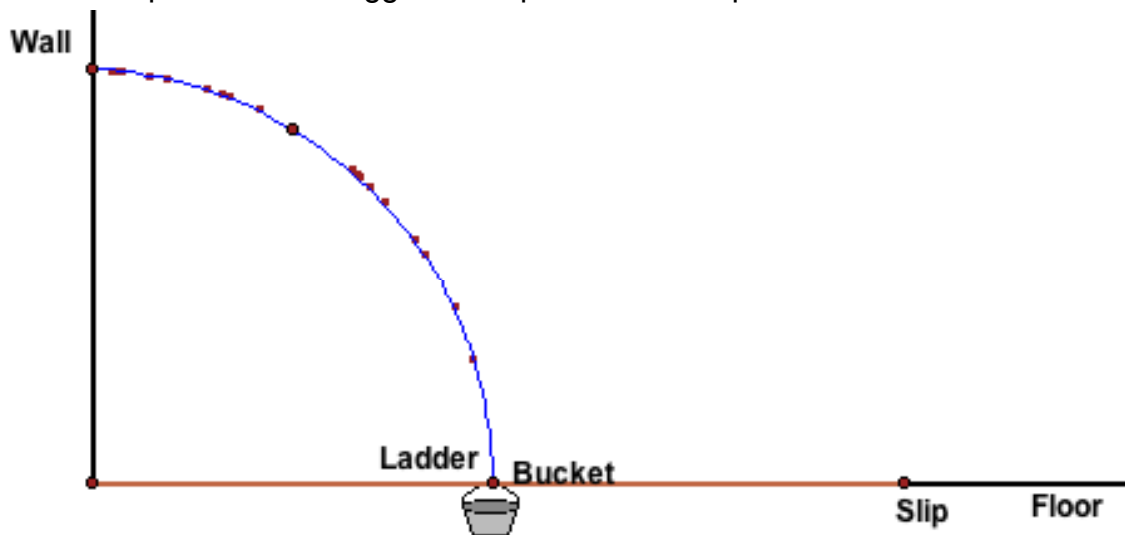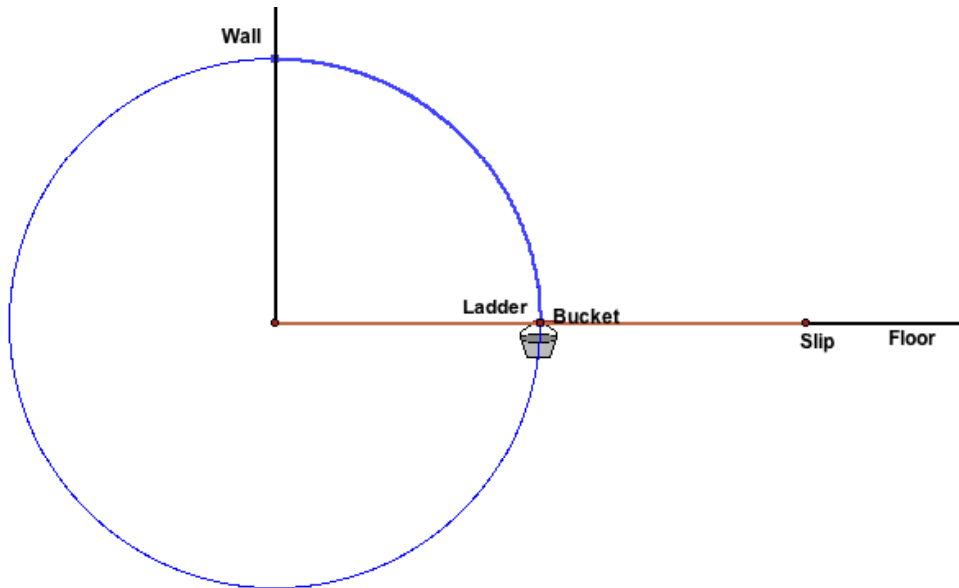

Mathematics used for Mathematical Focus 1

- Know the GSP calculus sample file exits and can be used for this experiment by plotting points over the trace
- Know that the arc through three or more points is unique

Mathematical Focus 2

We create a sample of data points using the Geometer's Sketchpad. Run the Falling Ladder.gsp simulation to create the path of the midpoint of the ladder. Construct a circle that passes through intersection of the floor and wall (point B in the given sketch) and the midpoint of the ladder when it is vertical horizontal. Since all of the data points lie on the circle and every data point from the midpoint of the ladder in its vertical position to the midpoint of the ladder in its horizontal position would form a continuous curve, the path is an arc.
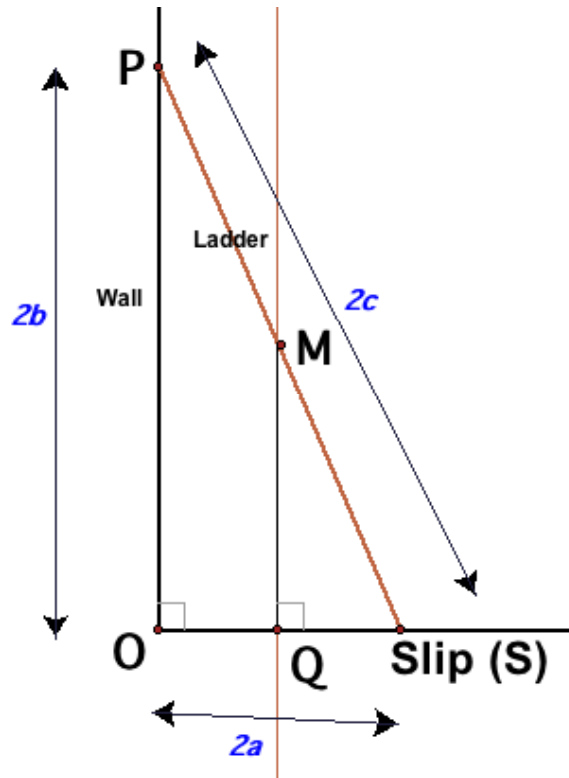
Mathematics used for Mathematical Focus 2

- Know the GSP calculus sample file exits and can be used for this experiment
- View arc as portion of a circle
- Know that there are infinitely many points on an arc

Mathematical Focus 3

The "Slip" (Point S) is at distance *2a* from O, the intersection of the floor and the wall. P is at distance *2b* from O. The length of the ladder is *2c*. Then the coordinates of S and P are (*2a*, 0) and (0, *2b*), respectively. The coordinates of M are (*a, b*).

Construct a line perpendicular to $\overline{OS}$ from M. The point where that line intersects $\overline{OS}$ is Q. $\triangle$OPS~$\triangle$QMS. So, $QM^2 + QS^2 = SM^2$ or $a^2 + b^2 = c^2$. The coordinates of any point M satisfy the equation of a circle with radius *c* and center (0,0). That is, the coordinates of M are of the form (m_1, m_2) where $m_1^2 + m_2^2 = c^2$. Since these coordinates could be any non-negative real numbers, the set of all possible points M would lie on an arc of the circle with center (0,0) and radius c that lies in the first quadrant. In addition, any point M is always equidistant from P, S, and O.

Mathematics used for Mathematical Focus 3

- Superimpose a coordinate system over a diagram or sketch.
- Choose coordinates to make calculations easy.
- Find coordinates of midpoint given coordinates of endpoints of a segment (or vice versa).
- Recognize a right triangle.
- Apply Pythagorean Theorem to describe relationship among lengths of sides of a right triangle.
- Recognize the equation of a circle (generated through the Pythagorean Theorem).
- Recognize similar triangles created by median line in right triangle.
- Know how to construct line through a point and perpendicular to a line.
- Interpret a variable used as a distance both as the distance between endpoint and midpoint and as the radius of a circle that is the distance from point on circle to the center of the circle.
- Generalize the relationship of distances from the midpoint to the endpoints and to the center of a circle to all possible midpoints.

Mathematical Focus 4

This is a geometry class. The physical situation of a ladder against the wall assumes that the ladder forms the hypotenuse of a right triangle and the legs of the right triangle lie along the wall and the ground at right angles:
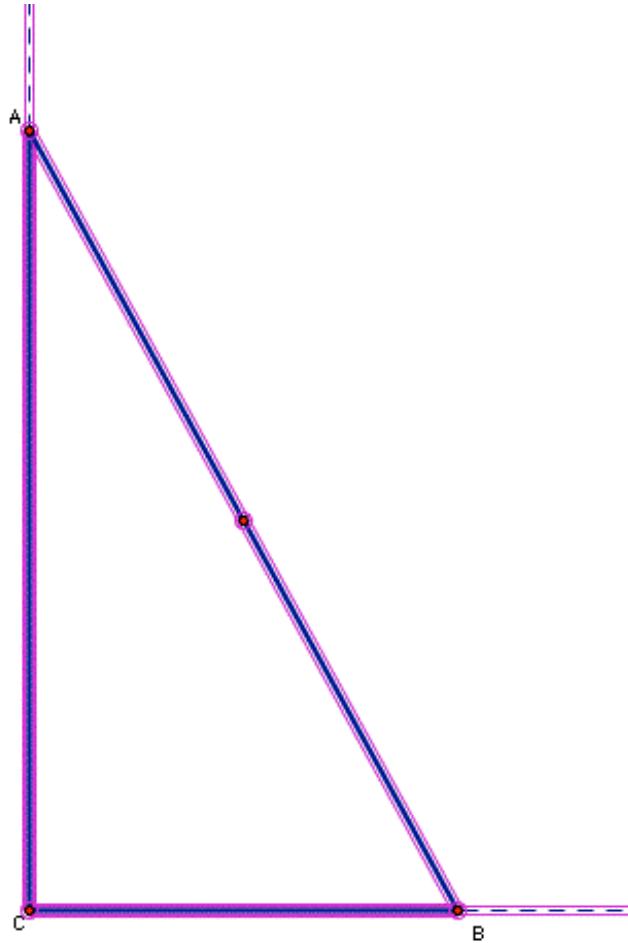
We have a question about the midpoint of the hypotenuse of a right triangle. It is a rather early theorem in Geometry:

THEOREM: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.

There are multiple elementary proofs of this theorem. However, the theorem would lead quickly to the following picture:
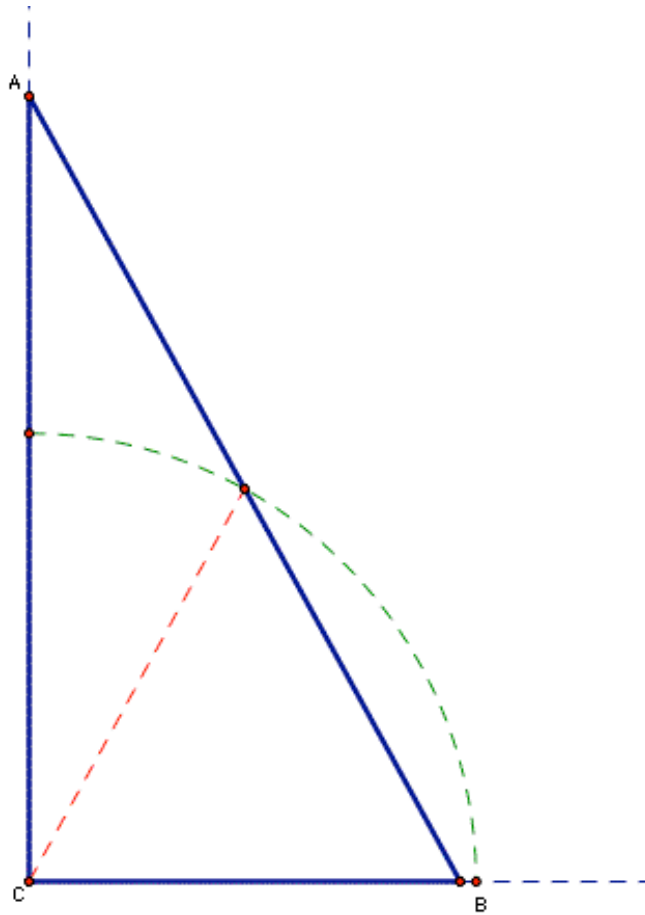
That is, as the ladder moves, the legs of the right triangle change but the hypotenuse (the ladder) remains the same length and the distance of its midpoint from the center will stay the same. The locus of the midpoint then lies along the arc of a circle with radius determined by the distance of the midpoint of the hypotenuse from the right angle vertex.

Mathematical Focus 5

In any approach to this problem, it is important to reinforce the idea that is an arc of a circle. In fact, the students should conclude that the arc is a quarter circle.

Mathematical Focus 6

If the problem is modified slightly to allow the fixed point on the ladder to be someplace other than the midpoint, the locus of the point will be an arc of one-quarter of an ellipse. The ideas in Mathematical Foci 1-3 would be applicable to explorations where student could observe and conclude this locus. Of course, the locus for the midpoint is an arc of a 'degenerate' ellipse (the arc of a circle).

References

Brown, Richard G., John W. Jurgensen, and Ray C. Jurgensen. *Geometry*. McDougal Littell, A Houghton Mifflin Company, Boston, MA: 2000. p. 405.