Situation 08: Ladder

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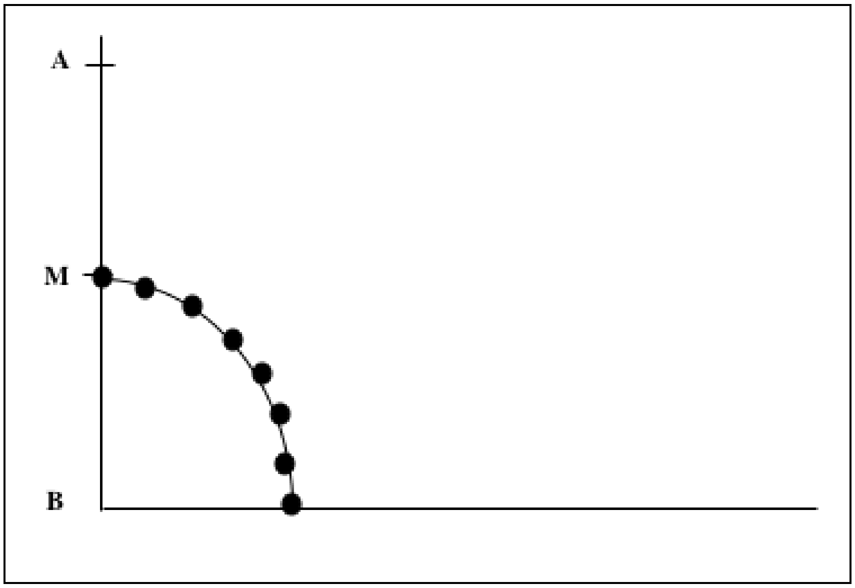
Prompt

A high school geometry class was in the middle of a series of lessons on loci. The teacher chose to discuss one of the homework problems from the previous day's assignment.

A student read the problem from the textbook (Brown, Jurgensen, & Jurgensen, 2000): *A ladder leans against a house. As A moves up or down on the wall, B moves along the ground. What path is followed by midpoint M? (Hint: Experiment with a meter stick, a wall, and the floor.)*

The teacher and two students conducted the experiment in front of the class, starting with a vertical "wall" and a horizontal "floor" and then marking several locations of M as the students moved the meter stick. The teacher connected the points. Their work produced the following data picture on the board:



A student commented, “That’s a heck of an arc.” Is it really an arc?

Commentary

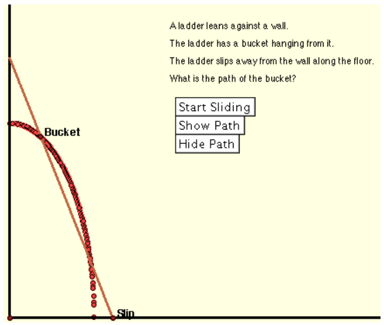
The foci for this situation show a variety of models and representations of the locus. In the class demonstration, the data was generated by a physical model.

Mathematical Foci

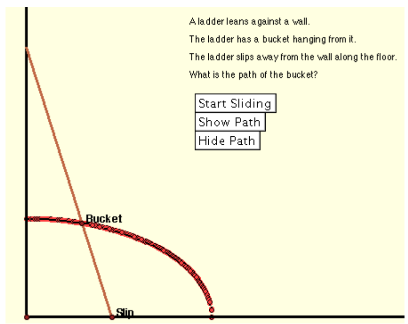
Mathematical Focus 1

Dynamic Geometry Systems, such as Geometer’s Sketchpad, can provide a powerful tool for exploring new situations.

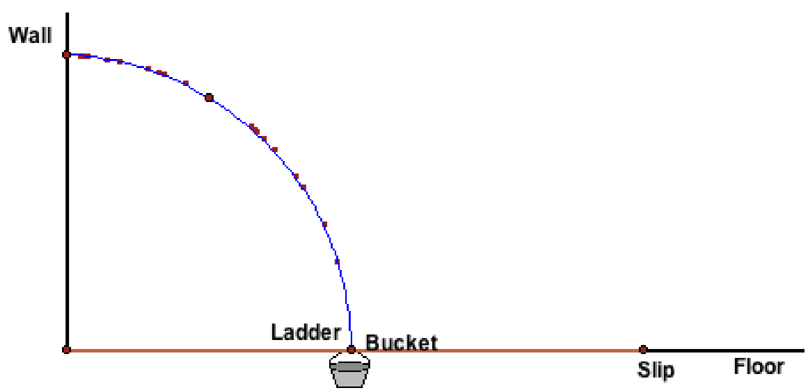
We create a sample of data points using the Geometer’s Sketchpad (VIG M 050606 Ladder arc.gsp based on GSP provided file, Falling Ladder.gsp). This simulation makes it easy to examine the locus of points created by the midpoint as it travels from the vertical to the horizontal position. It also makes it easy to examine the locus created by other points along the ladder. We can examine the locus of points when we consider a point above the midpoint of the ladder.



And the locus of points created by a point below the midpoint of the ladder



We can compare these the locus of points created by the midpoint of the ladder.

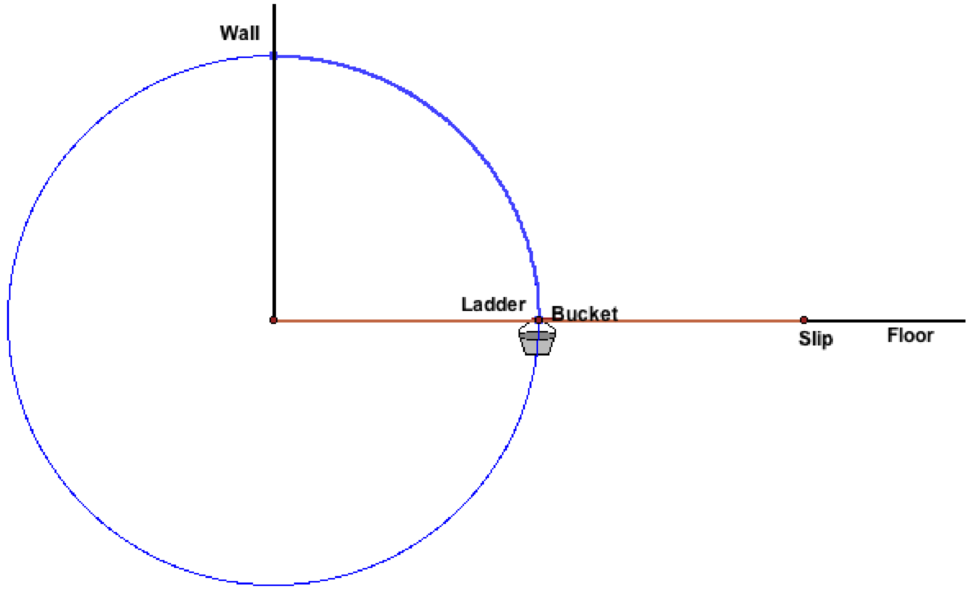


This comparison suggest that the shape is an arc because it is the shape to which the graphs converge.

Mathematical Focus 2

If all the points on the path created by the midpoint of the falling ladder lie on a circle then the path created is an arc of the circle.

We create a sample of data points using the Geometer's Sketchpad. Run the Falling Ladder.gsp simulation to create the path of the midpoint of the ladder. Construct a circle that passes through intersection of the floor and wall (point B in the given sketch) and the midpoint of the ladder when it is vertical horizontal. Since all of the data points lie on the circle and every data point from the midpoint of the ladder in its vertical position to the midpoint of the ladder in its horizontal position would form a continuous curve, the path is an arc.

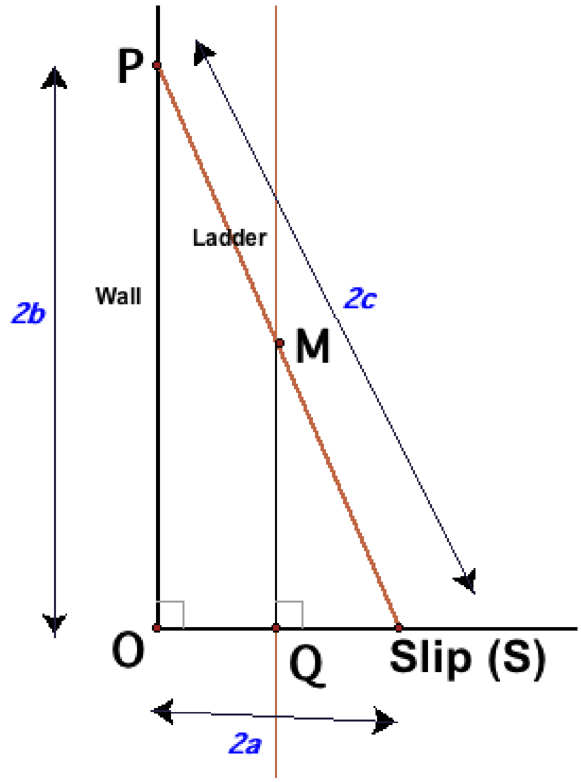


Mathematical Focus 3

A coordinate system can be superimposed over the sketch and coordinate geometry can be used to write an equation which relates the x and y coordinates of the ladder midpoint as it falls.

The "Slip" (Point S) is at distance *2a* from O, the intersection of the floor and the wall. P is at distance *2b* from O. The length of the ladder is *2c*. Then the coordinates of S and P are (*2a*, 0) and (0, *2b*), respectively. The coordinates of M are (*a*, *b*).

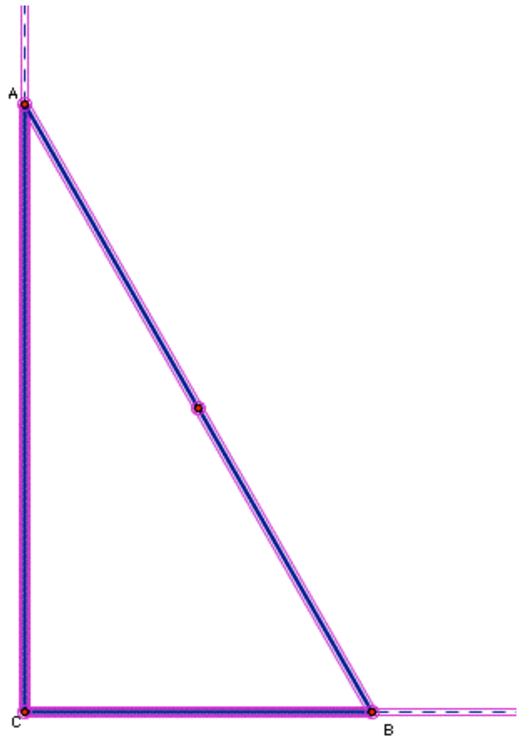
Construct a line perpendicular to $\overline{OS}$ from M. The point where that line intersects $\overline{OS}$ is Q. ΔOPS~ΔQMS. So, $\mathrm{QM}^2 + \mathrm{QS}^2 = \mathrm{SM}^2$ or $a^2 + b^2 = c^2$. The coordinates of any point M satisfy the equation of a circle with radius *c* and center (0,0). That is, the coordinates of M are of the form (m_1, m_2) where $m_1^2 + m_2^2 = c^2$. Since these coordinates could be any non-negative real numbers, the set of all possible points M would lie on an arc of the circle with center (0,0) and radius c that lies in the first quadrant. In addition, any point M is always equidistant from P, S, and O.



Mathematical Focus 4

The theorems and definitions of classic geometry can be used to demonstrate the locus of points is a circle.

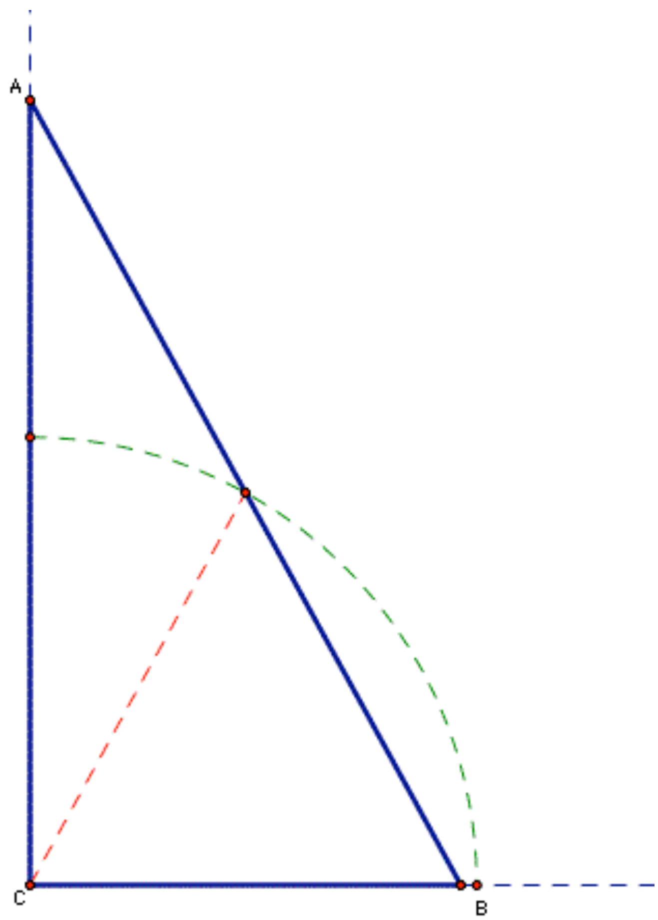
The physical situation of a ladder against the wall assumes that the ladder forms the hypotenuse of a right triangle and the legs of the right triangle lie along the wall and the ground at right angles:



We have a question about the midpoint of the hypotenuse of a right triangle. It is a rather early theorem in Geometry:

THEOREM: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.

There are multiple elementary proofs of this theorem. However, the theorem would lead quickly to the following picture:

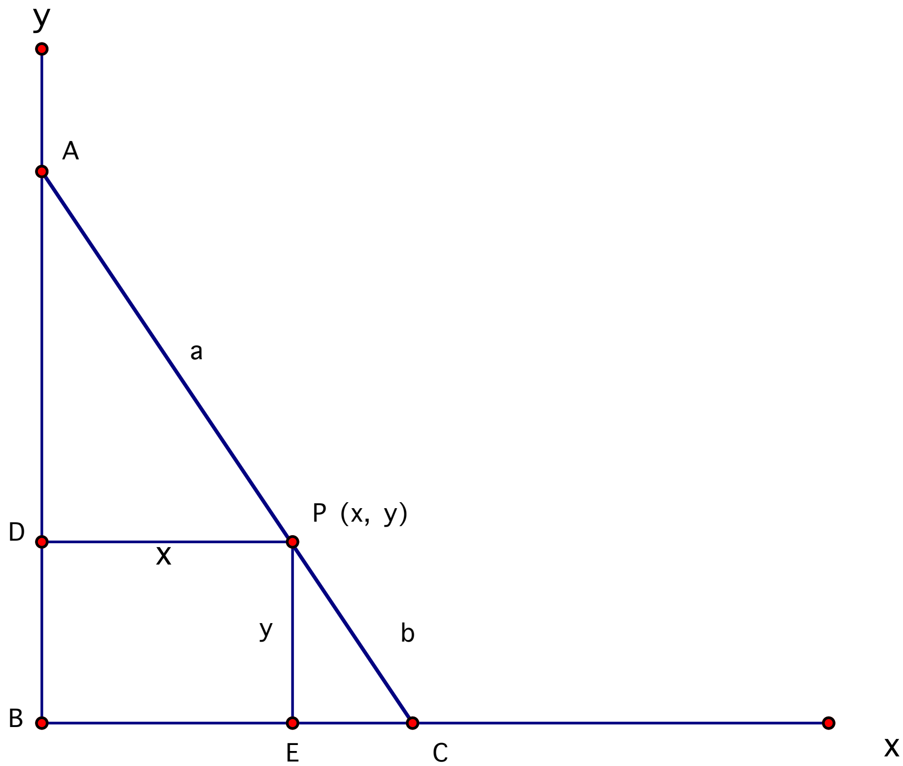


That is, as the ladder moves, the legs of the right triangle change but the hypotenuse (the ladder) remains the same length and the distance of its midpoint from the center will stay the same. The locus of the midpoint then lies along the arc of a circle with radius determined by the distance of the midpoint of the hypotenuse from the right angle vertex.

Mathematical Focus 5

Modifying the problem slightly to allow the fixed point on the ladder to be someplace other than the midpoint, produces a locus of points which is the arc of an ellipse.

Superimpose a rectangular coordinate system on the figure with the origin at the intersection of the wall and the ground. We can designate the coordinates of our fixed point as (x, y) and designate the distance from the top of ladder to our point as a and the distance from the bottom of the ladder to our point as b.



Because $\overline{DP}$ is parallel to $\overline{BC}$, $\triangle ADP \sim \triangle ABC$, $\dfrac{AP}{DP} = \dfrac{PC}{EC}$. By the Pythagorean Theorem, $m\overline{EC} = \sqrt{b^2 - y^2}$, so $\dfrac{a}{x} = \dfrac{b}{\sqrt{b^2 - y^2}}$ which yields $a\sqrt{b^2 - y^2} = xb \Rightarrow$

$$a^2(b^2 - y^2) = x^2b^2 \Rightarrow$$
$$a^2y^2 + x^2b^2 = a^2b^2 \Rightarrow$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of an ellipse with center at the origin and axes of length *2a* and *2b*. We can then examine the special case of our fixed point as the midpoint of the ladder which makes *a=b*. By substitution we find that the equation is now the equation of a circle with center at the origin and radius of a.

References

Brown, Richard G., John W. Jurgensen, and Ray C. Jurgensen. McDougal Littell, A Houghton Mifflin Company, Boston, MA: 2000. p. 405.