

Inverse Trig Functions

Prompt

Three prospective teachers have planned a unit of trigonometry as part of their work in a course on the teaching and learning of secondary mathematics. They have developed a plan in which students first encounter what they call “the three basic functions”: sine, cosine, and tangent. They have indicated in their plan that they would next have students work with “the inverse functions,” apparently meaning the secant, cosecant, and cotangent.

How might the instructor respond to this proposal?

Mathematical Analysis

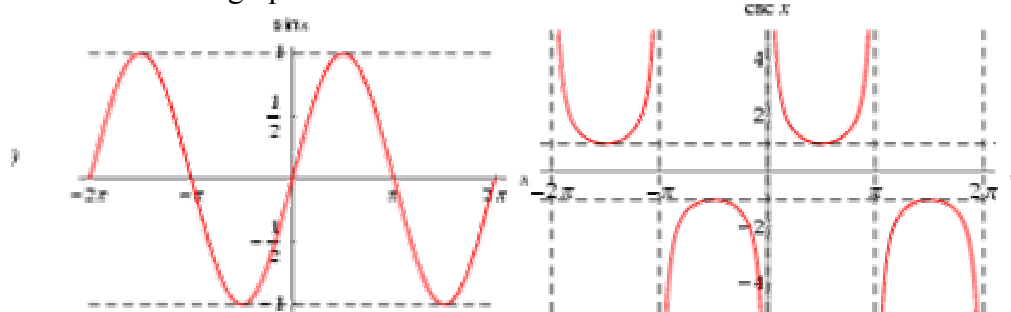
Crux. There appears to be some confusion between *inverse function* and *multiplicative inverse* (reciprocal); in other words, there’s ambiguity in the use of the term *inverse* (for functions and for operations). In the former case, the elements are functions and the operation is composition; in the latter case, the elements are numbers and the operation is multiplication.

Concepts and notations. The functions *secant*, *cosecant*, and *cotangent* are defined, respectively, as follows: $\sec x \equiv \frac{1}{\cos x}$, $\csc x \equiv \frac{1}{\sin x}$, and $\cot x \equiv \frac{1}{\tan x}$. They are not defined for $\cos x = 0$, $\sin x = 0$, and $\tan x = 0$.

In contrast, the *inverse trigonometric functions* of the trigonometric functions *sine*, *cosine*, and *tangent* are the following, respectively: $\sin^{-1} x$ (sometimes written $\arcsin x$), $\cos^{-1} x$ (sometimes $\arccos x$), and $\tan^{-1} x$ (sometimes $\arctan x$ or $\text{arctg } x$). In general, given a function $f(x)$, its inverse $f^{-1}(x)$ is defined by $f(f^{-1}(x)) = f^{-1}(f(x)) \equiv x$. Part of the confusion in the vignette, therefore, may stem from the unfortunate notation $f^{-1}x$, which conflicts with the common interpretation of a superscripted quantity as indicating a power: $f^{-1}x = (1/f)x = x/f$.

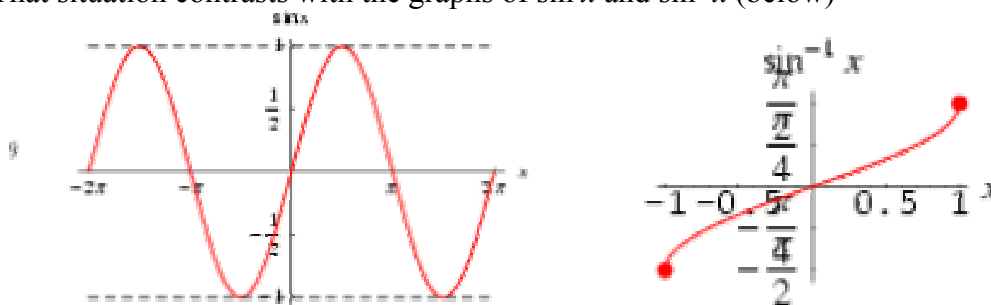
The inverse trigonometric functions are *multivalued*. For example, there are multiple values of x such that $y = \sin x$, so $\sin^{-1} x$ is not uniquely defined unless a *principal value* is defined (by restricting the domain). For example, $\sin^{-1} x$ is typically defined only for $-1 \leq x \leq 1$. Principal values are sometimes denoted with a capital letter; so, for example, the principal value of the *inverse sine* $\sin^{-1} x$ may be denoted $\text{Sin}^{-1} x$ or $\text{Arcsin } x$ (but this capitalization notation is far from universal and may, in fact, be used with the opposite meaning).

Representations. The graphs of $\sin x$ and $\csc x$ are shown below:



When they are superimposed, one can see that $(\sin x)(\csc x) = 1$. When $\sin x$ approaches 1, so does $\csc x$. When $\sin x$ approaches 0, $\csc x$ approaches ∞ .

That situation contrasts with the graphs of $\sin x$ and $\sin^{-1}x$ (below)



When the principal value of $\sin x$ (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) is superimposed on the graph of $\sin^{-1}x$, one can see that the two functions are reflections about the line $y = x$, which is the situation for any pair of inverse functions.

Related (simpler) problems. Consider the linear function $f(x) = \frac{1}{3}x$, which passes through the origin. Its inverse is $f^{-1}(x) = 3x$, which is another line through the origin and reflected in $y = x$. In contrast, the function $g(x)$ obtained by taking the reciprocal of $f(x)$ is $g(x) = \frac{3}{x}$, which is a hyperbola centered at the origin. The two graphs intersect at $(3, 1)$, and when they are superimposed, one can see that their product is 1. Similarly, if $h(x) = x^2$, its inverse is \sqrt{x} , which has to be restricted to nonnegative x , and its reciprocal is $1/x^2$, which is a cubic function.

History and uses. Tables of trigonometric functions were constructed by ancient Greek astronomers in terms of ratios of arcs and chords of circles, and the sine function was defined in ancient India as the length of the side opposite an angle in a right triangle with a fixed hypotenuse. In the 16th century, the German astronomer Georges Joachim introduced the modern conception of trigonometric functions as ratios instead of the lengths of certain lines or the arcs of a circle. Because there were three sides in a right triangle, there were six possible ratios, and because calculation with them was so difficult in the days before calculators, names were given to each one and tables constructed for them. The advent of computers has meant that given one of the functions, the others are easily calculated (using trigonometric identities). The secant, cosecant, and cotangent, which were never used much in applications, have consequently diminished somewhat in importance relative to the other three.

Although the trigonometric functions were originally defined for angles with measures between 0° and 180° and used to solve problems of indirect measurement, applications involving the modeling of periodic phenomena required their extension first to all real arguments and eventually to the domain of complex numbers. Modern textbooks sometimes use a unit circle definition for the trigonometric functions and sometimes use a wrapping-function definition.

The inverse trigonometric functions—especially the inverse sine, inverse tangent, and inverse secant—turn out to be useful in calculus as antiderivatives for integrals involving quotients and roots of polynomials. They are often used in precalculus courses primarily to illustrate the concept of inverse function.