Main Ideas for "Situation 3" from Penn State

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Key Question: What are the main mathematical ideas that the methods course instructor might consider in order to address this issue[1]?

Main Ideas:

- The fundamental mathematical issue seems to be the notion of an inverse, and the different kinds of inverses that are common in mathematical thinking (e.g., additive, multiplicative, and functional inverses). To address the situation, I believe the teacher needs to consider what kinds of thinking might be required to produce and operate with these different these different kinds of inverses and what these particular students' conceptions of these different kinds of inverses might be.

- Since these students are prospective teachers, they have undoubtedly experienced—and are likely very comfortable with—additive and multiplicative inverses of numbers, quantities, and algebraic expressions. For example, I would guess that they can state the additive inverse, or the multiplicative inverse, of a number, quantity, or algebraic expression and use both kinds of inverses in solving equations (e.g., add the additive inverse of $5x$ to both sides of an equation, or multiply both sides of an equation by the multiplicative inverse of -3). Still, the teacher may want to work on notions of multiplicative inverses of algebraic expressions with the students (e.g., what are the multiplicative inverses of $3x$, $x + 5$, $x^2 + 3x + 2$, etc.) There are a host of issues to discuss—i.e., what the multiplicative inverse of an expression can mean, domain of x, the notion of a reciprocal, etc. This work can lay more ground for defining, e.g., sec(x) as the reciprocal of cos(x), etc. (which is more general than just defining sec(π /3) as the reciprocal of cos (π/3).

- However, I would guess that the students may not be as comfortable with thinking about additive and multiplicative inverses of a set of function values (which are not usually discussed, to my knowledge). That is, what is the additive inverse of the function $f(x) = x^2 + 5x + 6$? What is the multiplicative inverse of $g(x) = x + 1$? Even though the students have certainly worked with functions like $h(x) = -x^2 – 5x – 6$ and $m(x) = 1/(x + 1)$ before, they may not have thought about this notion of taking an additive inverse or a multiplicative inverse of a set of numbers or quantities that are varying across all possible values of x. (Does this sound crazy?) The reason I bring this up, even though it is unconventional, is that of course there is a sense in which the secant function is an inverse of the cosine function (since each the secant of each value of x is a multiplicative inverse of the cosine at that value), and the teacher may want the students to see how they are "right" in thinking about inverses in this context. In my experience teaching high school precalculus students, the first time they see functions as multiplicative inverses in

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[1] Note that "this issue" can be a question from a student—i.e., something a student recognizes as problematic—or it can be, as in this case, something that the *teacher* recognizes as problematic and the student does not.

this way is with the trig functions. That is, we do not too often work on y = x + 1 in conjunction with rational functions like y = 1/(x + 1)—usually linear functions and rational functions are separated by much other algebraic ground. In contrast, precalculus students usually work on y = cos(x) and y = sec(x) closely together, generally not long after defining secant as the multiplicative inverse, or reciprocal, of cosine.

- Finally, I believe the teacher needs to assess these students' understanding of *inverse functions* (which may entail assessing their understanding of functions and function composition). This assessment may be particularly critical with trigonometric functions, which are often relatively more mysterious for students than linear or quadratic functions, for example. There are many ideas that are important to consider in thinking about inverses:

  - The notion of undoing: That is, what does it mean to undo a mathematical process? How do you "undo" taking the sine of an angle (or number)? (This idea lends itself more to thinking about individual values of a function than to consider the function as a whole "entity" or object, but it's still important.)

  - The notion of switching all ordered pairs in a function (i.e., x-values become y-values, and y-values become x-values). This idea can grow out of the undoing (i.e., output becomes the input in order to undo a process) and can lead to considering the relationship between the graphs of a function and its inverse (and to determining whether the inverse of a function is a function). What does it mean to switch all of the x and y values of the sine function? Is this a new function? (Only if the domain is restricted. The teacher would need to remind students that this new relation **is the inverse of the sine function**, called arcsin(x), but it is only a function, Arcsin(x), on the restricted domain, $-\pi/2$ to $\pi/2$, I believe.) How is it related to the cosecant function? What does it mean to find the inverse of the cosecant function?

  - The notion of composing mathematical processes (what happens if you do a mathematical process and then undo it? You get back where you started). So what happens if you first take the sine of any but no particular value of x and then undo it by "un-sining" it (i.e., taking the inverse sine of it)? You get back "the" value of x you started with (which was any but no particular value!).

  - One caveat in all this…students may very well need to work more with the notion of trigonometric *functions* (or even functions more generally) before they are ready for working on inverse functions…don't know if this is going to far afield.

A note about pathways and main ideas:

It seems to me that the main ideas the teacher calls upon in deciding how to address a given issue has a lot to do with her knowledge of the students' mathematical thinking.

- For example, if the methods course instructor has inferred that the work of these prospective teachers is quite strong mathematically (i.e., she has seen them work competently with functions and perhaps even inverses previous to this point), then she might see their characterization of secant, cosecant, cotangent as inverse functions to be a minor conflation. Brief work with vocabulary and sorting out the different meanings of inverses that the students held and that are commonly accepted in standard mathematical discourses would likely be sufficient to address the issue.

- If, on the other hand, the methods course instructor had inferred that the work of these prospective teachers was weak in certain areas (e.g., she had seen them have difficulty with the concept of function, etc.), then she might see their conflation as more significant. Addressing it might require more extensive work with the notion of function and covariation, and in particular with the notion of a trigonometric function, before the notion of inverse functions could be examined.

- If the methods course instructor was not very certain about the prospective teachers' current mathematical thinking (i.e., let's say that this incident happened quite early in the term), then she would likely need to "make tests" about the students' conceptions of inverses, functions. and trig functions, before determining what else to do.