MAC-CPTM Situations Project

Situation 03: Inverse Trigonometric Functions

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Prompt

Three prospective teachers planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which high school students first encounter what the prospective teachers called "the three basic trig functions": sine, cosine, and tangent. The prospective teachers indicated in their plan that students next would work with "the inverse functions," identified as secant, cosecant, and cotangent.

Commentary

The problem seems centered on understanding the entity of inverse. The foci draw on the general notion of inverse from collegiate mathematics and the use of inverse in school mathematics. The appropriate use of inverse in school mathematics depends on a solid understanding of the concept at the collegiate level. When we think about inverses in that context, we need to think about the operation, the set of elements on which the operation is defined, and the identity element given this operation and set of elements. Anchoring school mathematics routines involving inverse functions in the definition of inverse function helps provide a foundation for those routines. For example, deriving a rule for an inverse function by reflecting the graph of the function over $y = x$ requires examination of the relationship between ordered pairs (a,b) and (b,a). The superscript, -1, can be used to identify inverse functions in general instead of merely as a way to specify a reciprocal.

Mathematical Foci

Mathematical Focus 1

An inverse requires three entities: a set, an operation defined on that set, and an identity element given this operation and set of elements.

The problem seems centered on knowing about the mathematical entity of inverse. An inverse requires three elements: a set, an operation defined on that set, and an identity element for the algebra consisting of these elements and operation. In general, if * is the operation, S is the set of elements on which * is defined, and $e \in S$ and e is the identity element, then for $c, d \in S$, c is an inverse of d in the system if and only if $c*d=d*c=e$. In the given trigonometry case, we are assuming the set is $\Re \to \Re$ functions, the operation is composition of functions, and the identity is given by f(x) = x. An observation such as csc(sin(0))≠0 is enough to conclude csc(x) is not an inverse function under composition for sin(x). However, if we change the mathematical circumstance to multiplication of real numbers with multiplicative inverse 1, for any value of x such that csc(x) $\neq$ 0, the number csc(x) is the multiplicative inverse for the number, sin(x). If function g is an inverse of a the sine function then g(sin(x))=sin(g(x)).

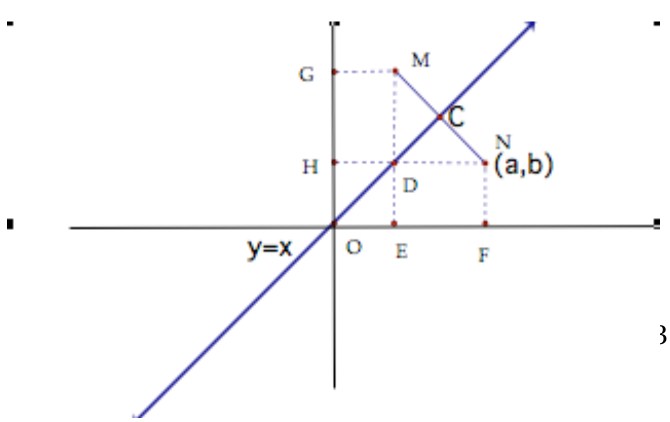
Secondary mathematics involves work with many different contexts for inverses. For example, opposites are additive inverses defined for real numbers and with additive identity of 0 and reciprocals are multiplicative inverses defined for non-zero real numbers and with multiplicative identity of 1.

Mathematical Focus 2
When functions are graphed in an x-y coordinate system with y as a function of x, inverses of functions (under composition) are the reflections of each other over the line y = x.

When functions are graphed in an x-y coordinate system with y as a function of x, inverses of functions are the reflections of each other over the line $y = x$. Justifying this claim requires establishing that (1) the reflection of an arbitrary ordered pair (a, b) over the line $y = x$ is the ordered pair (b, a), and (2) the functions f and g are inverses if and only if g consists of the ordered pairs (f(x), x) and f consists of the ordered pairs (g(x), x).

A geometric argument can be used to show that the reflection of an arbitrary ordered pair (a, b) over the line $y = x$ is the ordered pair (b, a), using the geometric properties of reflection (A' is a reflection of A over line l if and only if l is the perpendicular bisector of $\overline{AA'}$). It can be shown that, if (x_0,y_0) is the reflection of (a, b) over $y = x$, then $x_0=b$ and $y_0=a$. One such argument is the following.

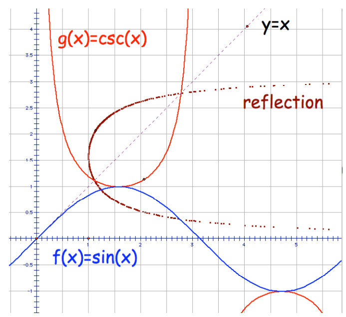
Reflect N = (a,b) over the line that is the graph of y = x. Call the image of N under this reflection M. Drop perpendiculars from M and N to the x-axis and to the y-axis. Call the points of intersect



ion E, F, G, and H. $\triangle MCD \cong \triangle NCD$ by SAS. MD=ND since corresponding parts of congruent triangles are congruent. DE=DH since D lies on the line y=x. ME = MD + DE = ND + DH = *a*. That is the y-coordinate of M is *a*.

$\overleftrightarrow{NF} \parallel \overleftrightarrow{ME} \parallel y-axis$ and $\overleftrightarrow{MG} \parallel \overleftrightarrow{NH} \parallel x-axis$. Since the x-y coordinate system is a rectangular coordinate system, the x-axis is perpendicular to the y-axis. So $\overleftrightarrow{ME} \perp x-axis$, $\overleftrightarrow{NF} \perp x-axis$, $\overleftrightarrow{MG} \perp y-axis$, and $\overleftrightarrow{NH} \perp y-axis$. Thus DNFE is a rectangle and MDHG is a rectangle. Thus GM = DH and NF = DE. Since DE = DH, GM = NF = *b*. That is, the x-coordinate of M is *b*. So the reflection of (*a*,*b*) over the line y=x is (*b*,*a*).

Support for the conclusion that the functions *f* and *g* are inverses if and only if *g* consists of the ordered pairs (*f*(*x*), *x*) and *f* consists of the ordered pairs (*g*(*x*), *x*) comes from consideration of domain and range issues. In order for f and g to be inverse functions, *f*(*g*(*x*))=*g*(*f*(*x*))=*x*. The range of *f* must be the domain of *g* and the range of *g* must be the domain of *f*. This is equivalent to saying *g* consists of the ordered pairs (*f*(*x*), *x*) and *f* consists of the ordered pairs (*g*(*x*), *x*).



Suppose cosecant and sine are inverse functions. A reflection of the graph of $y = \csc(x)$ in the line $y = x$ would be the graph of $y = \sin(x)$. The figure at the right shows, on one coordinate system, graphs of the sine function, the line given by $y = x$, the cosecant function, and the reflection in $y = x$ of the cosecant function. Because the reflection and the sine function graph do not coincide, sine and cosecant are not inverse functions.

It is possible to choose a subset, S, of $\Re$ such that and so that we can define function, *g*, with S as its domain such that *g(x)*=sin(*x*).

Mathematical Focus 3

The notation f^{-1}, used to show the inverse of f in function notation, involves only one of several mathematically valid interpretations of the superscript, -1.

The notation f^{-1} is often used to show the inverse of *f* under composition in function notation. When working with rational numbers, x^{-1} is used to represent the reciprocal of *x*. If people think about the "inverse of sine" as $\sin^{-1}$, they might use $\frac{1}{\sin(x)}$ to represent the inverse of sine under composition.