Situation 3: Inverse Trigonometric Functions

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18 June 2005 – RMZ and MKH

Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
25 July 2006 – Ryan Fox

Prompt

Three prospective teachers have planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which students first encounter what they call the three basic functions: sine, cosine, and tangent. They indicated in their plan they would next have students work with the inverse functions apparently meaning the secant, cosecant, and cotangent functions.

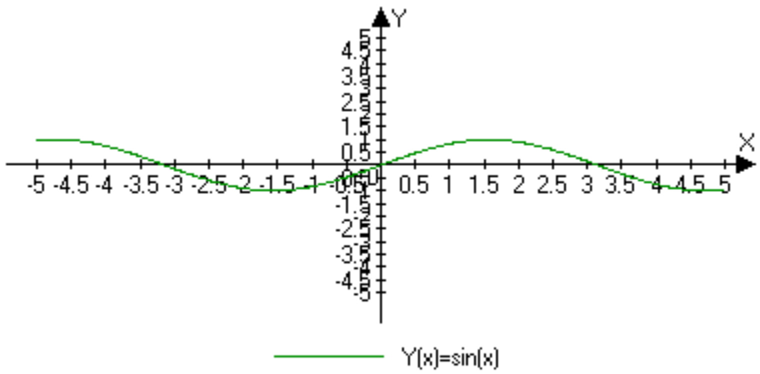
Commentary

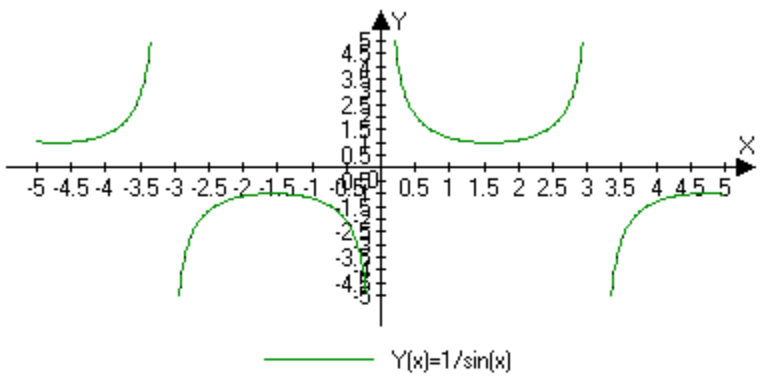
The problem seems centered on knowing about the entity of inverse. Connections can be made to the notion of inverse from abstract algebra. When we think about inverses, we need to think about the operation and the elements on which the operation is defined. The selection of foci is made to emphasize the difference between an inverse for the operations of multiplication and composition of functions. Each focus contrasts how the multiplicative inverse invalidates the properties for an inverse element for the operation of composition. The contrasts will be illustrated in a variety of approaches: graphical, numerical, and verbal.

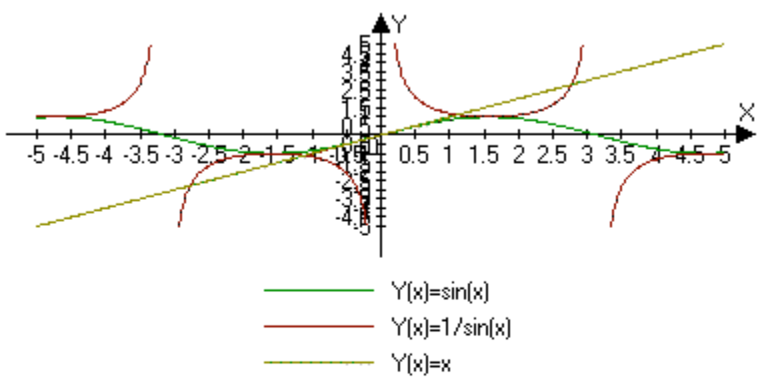
Mathematical Foci

Mathematical Focus 1

This is an example of a contradiction that might result from the manifestation in this scenario of the students' concept of inverse function. Suppose cosecant is the inverse function for the sine function. Then wouldn't it be true that the reflection of sin (x) over the line y = x be csc (x)? Let's look at the graphs of f(x) = sin x and g(x) = csc (x).







The inverse of a function can be found by switching the sets for domain and range. This can be represented graphically by switching the *x* and *y* values of all the coordinates of the original graph. When the coordinates are swapped, the graph of the line *y* = *x* acts as a line of reflection. Notice that csc (x) is not the inverse of sin (x) since the graph of csc (x) is not the reflection of the graph of sin (x) over the line y = x.

Mathematical Focus 2

The lack of rigor in the application of the terminology inverse has created a conceptual misunderstanding in the minds of the students. The students have confused the inverse under multiplication and functional composition. The students might understand the difference between the inverses under addition and multiplication for the same element. Generally, an inverse is an element of a set that when it is combined with a second element yields the identity element. For example, the additive inverse of 2 is -2 because 2 + (-2) = 0 and the multiplicative inverse of 2 is ½ since 2 x ½ = 1 . Both -2 and ½ are inverses of 2, but the choice of operation determines which element is the appropriate inverse.

Working back to the trig functions, one might think of csc (x) as an inverse of sin (x) because csc (x) is the reciprocal of sin (x). For any value of x such that csc (x) does not equal 0, the number csc (x) is the multiplicative inverse for the number, sin (x): sin (x) x csc (x) = 1. However, when we are looking for an inverse function, the elements are functions and the operation is composition. The goal of an inverse under composition of functions is to yield x. It is not always the case that sin (csc x) = x, then we can not say that the two functions are inverses under the composition of functions.

Mathematical Focus 3

Two functions $f(x)$ and $g(x)$ are inverses of each other if and only if $f(g(x)) = g(f(x)) = x$. It can be shown through this definition that sin x and csc x are not inverses of each other, using one value to show that this formula does not hold.

$$\sin\left(\csc\frac{\pi}{4}\right) = \sin 1 \approx 0.8415$$

Since $\frac{\pi}{4} \neq .8415$, then it is the case that these two functions are not inverse of each other.