MAC-CPTM Situations Project

Situation 03: Inverse Trigonometric Functions

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Prompt

Three prospective teachers planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which high school students would first encounter what the prospective teachers called "the three basic trig functions": sine, cosine, and tangent. The prospective teachers indicated in their plan that students next would work with "the inverse functions," identified as secant, cosecant, and cotangent.

Commentary

The inverse is central to the set of mathematical foci that follow. The foci draw on the general concept of inverse and its use in school mathematics. Key ideas related to the inverse are the operation being inverted, the set of elements on which the operation is defined, and the identity element given this operation and set of elements. The notation is also a key idea: f^{-1}, used to show the inverse of f in function notation, involves only one of several mathematically valid interpretations of the superscript -1. Anchoring in the definition of inverse function school mathematics routines that involve inverse functions helps provide a foundation for those routines. For example, deriving a rule for an inverse function by reflecting the graph of the function in *y* = *x* requires an examination of the relationship between ordered pairs (*a, b*) and (*b, a*).

Mathematical Foci

Mathematical Focus 1

An inverse requires three entities: a set, an operation defined on that set, and an identity element given this operation and set of elements.

Secondary mathematics involves work with many different contexts for inverses. For example, opposites are additive inverses defined for real numbers and with additive identity of 0, and reciprocals are multiplicative inverses defined for non-zero real numbers and with multiplicative identity of 1.

In general, if * is the operation, *S* is the set of elements on which * is defined, $e \in S$, and *e* is the identity element, then for $c, d \in S$, *c* is an inverse of *d* in the system if and only if $c*d = d*c = e$. In the given case of trigonometric functions, the set is assumed to be $\Re \rightarrow \Re$ functions, the operation is composition of functions, and the identity is given by *f*(*x*) = *x*. An observation such as csc(sin(0)) ≠ 0 is sufficient to show that csc(*x*) is not an inverse function under composition for sin(*x*). In the case of multiplication of real numbers, in contrast, the multiplicative inverse is 1, and for any value of *x* such that csc(*x*) ≠ 0, the number csc(*x*) is the multiplicative inverse for the number sin(*x*). If function *g* is the inverse of the sine function, then *g*(sin(*x*))=sin(*g*(*x*)).

Mathematical Focus 2

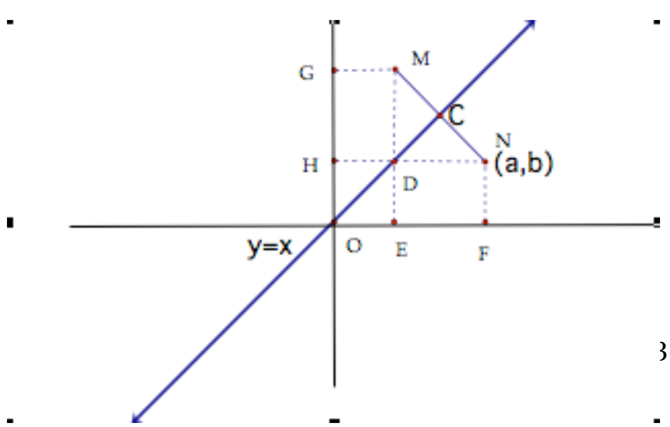
When functions are graphed in an xy*-coordinate system with* y *as a function of* x*, inverses of functions (under composition) are the reflections of each other over the line* y = x.

The graph of the image of a function reflected over the line *y* = *x* is the graph of its inverse function. Justifying this claim requires establishing that (1) the reflection of an arbitrary ordered pair (*a*, *b*) over the line *y* = *x* is the ordered pair (*b*, *a*), and (2) the functions *f* and *g* are inverses if and only if *g* consists of the ordered pairs (*f*(*x*), *x*) and *f* consists of the ordered pairs (*g*(*x*), *x*).

The following geometric argument shows that the reflection of an arbitrary ordered pair (*a*, *b*) in the line *y* = *x* is the ordered pair (*b*, *a*), using the geometric properties of reflection (A' is a reflection of A in line *l* if and only if *l* is the perpendicular bisector of $\overline{AA'}$). It can be shown that if (x_0, y_0) is the reflection of (*a*, *b*) in *y* = *x*, then $x_0 = b$ and $y_0 = a$.

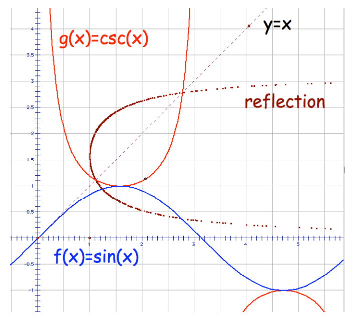
Reflect N = (*a*, *b*) in the line that is the graph of *y* = *x*.

Call the image of N under this reflection M. Since the *xy*-coordinate system is a rectangular coordinate system, it



is possible to drop perpendiculars from M and N to the x-axis and to the y-axis. Call the points of intersection E, F, G, and H. ∠MCD ≅ ∠NCD, so it follows that MD = ND. $\overleftrightarrow{NF} \parallel \overleftrightarrow{ME} \parallel y - axis$ and $\overleftrightarrow{MG} \parallel \overleftrightarrow{NH} \parallel x - axis$. Also, the x-axis is perpendicular to the y-axis. So $\overleftrightarrow{ME} \perp x - axis$, $\overleftrightarrow{NF} \perp x - axis$, $\overleftrightarrow{MG} \perp y - axis$, and $\overleftrightarrow{NH} \perp y - axis$. Therefore, ΔDEO ≅ ΔDHO, and it follows that DE = DH. ME = MD + DE = ND + DH = a. That is, the y-coordinate of M is a. Thus, DNFE is a rectangle, and MDHG is a rectangle. Hence, GM = DH and NF = DE. Since DE = DH, GM = NF = b. That is, the x-coordinate of M is b. So the reflection of (a, b) in the line $y = x$ is (b, a).

Suppose cosecant and sine are inverse functions. A reflection of the graph of $y = \csc(x)$ in the line $y = x$ would be the graph of $y = \sin(x)$. The figure at the right shows, on one coordinate system, graphs of the sine function, the line given by $y = x$, the cosecant function, and the reflection about $y = x$ of the cosecant function. Because the reflection of the graphs of the cosecant function and the sine function do not coincide, sine and cosecant are not inverse functions.



Support for the conclusion that the functions f and g are inverses if and only if g consists of the ordered pairs $(f(x), x)$ and f consists of the ordered pairs $(g(x), x)$ comes from consideration of issues of domain and range. In order for f and g to be inverse functions, $f(g(x)) = g(f(x)) = x$. The range of f must be the domain of g, and the range of g must be the domain of f. This is equivalent to saying that g consists of the ordered pairs $(f(x), x)$, and f consists of the ordered pairs $(g(x), x)$.

Let $f(x) = \sin x$ and $g(x) = \csc x$.

x	$f(x) = sin\ x$	$g(f(x)) = csc\ (sin(x))$
0	0	#DIV/0!
0.523598776	0.5	2.085829643
0.785398163	0.707106781	1.539321334
1.047197551	0.866025404	1.312749454
1.570796327	1	1.188395106

The columns for x and $g(f(x))$ show that $g(f(x)) \neq x$, so these two functions do not satisfy this property of inverse functions.