MAC-CPTM Situations Project

Situation 03: Inverse Trigonometric Functions

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Prompt

Three prospective teachers planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which high school students would first encounter what the prospective teachers called “the three basic trig functions”: sine, cosine, and tangent. The prospective teachers indicated in their plan that students next would work with “the inverse functions,” identified as secant, cosecant, and cotangent.

Commentary

The foci draw on the general concept of inverse and its multiple uses in school mathematics. Key ideas related to the inverse are the operation involved, the set of elements on which the operation is defined, and the identity element given this operation and set of elements. The crux of the issue raised by the prompt lies in the use of the term *inverse* with both functions and operations.

Mathematical Foci

Mathematical Focus 1

An inverse requires three entities: a set, an operation defined on that set, and an identity element given this operation and set of elements.

Secondary mathematics involves work with many different contexts for inverses. For example, opposites are additive inverses defined for real numbers and with additive identity of 0, and reciprocals are multiplicative inverses defined for non-zero real numbers and with multiplicative identity of 1.

In general, if * is the operation, S is the set of elements on which * is defined, $e \in S$, and e is the identity element, then for $c, d \in S$, c is an inverse of d in the system if and only if $c*d = d*c = e$.

In functions, including trigonometric functions, the set is assumed to be $\Re \rightarrow \Re$ functions, the operation is composition of functions, and the identity is given by $f(x) = x$. If a function g is the inverse of the sine function, for example, then $g(\sin(x))=\sin(g(x)) = x$. An observation such as $\csc(\sin(0)) \neq 0$ is sufficient to show that $\csc(x)$ is not an inverse function under composition for $\sin(x)$. In the case of multiplication of real numbers, in contrast, the multiplicative inverse is 1, and for any value of x such that $\csc(x) \neq 0$, the number $\csc(x)$ is the multiplicative inverse for the number $\sin(x)$.

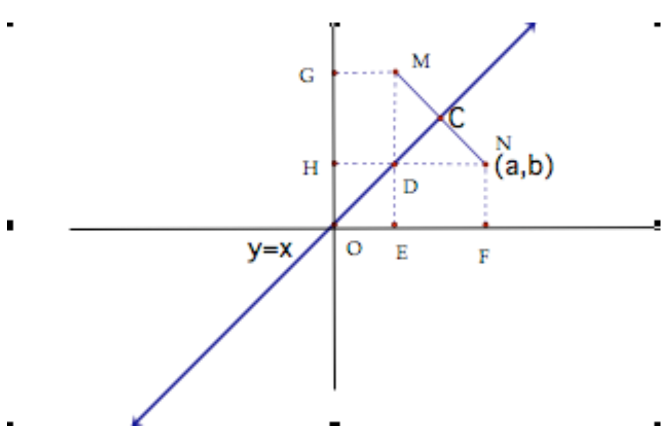
Mathematical Focus 2

When functions are graphed in an xy*-coordinate system with* y *as a function of* x*, the graphs of inverse functions (under composition) are images under the reflection in the line* y = x.

The graph of a function reflected in the line $y = x$ is the graph of its inverse function. Justifying this claim requires establishing that (1) the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a), and (2) the functions f and g are inverses if and only if g consists of the ordered pairs $(f(x), x)$ and f consists of the ordered pairs $(g(x), x)$.

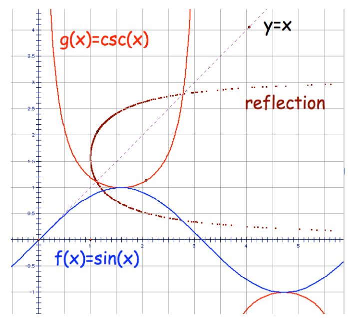
The following geometric argument shows that the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a), using the geometric properties of reflection (A' is a reflection of A in line l if and only if l is the perpendicular bisector of $\overline{AA'}$). It can be shown that if (x_0, y_0) is the reflection of (a, b) in $y = x$, then $x_0=b$ and $y_0 =a$.

Reflect N = (a, b) in the line that is the graph of $y = x$. Call the image of N under this reflection M. Since the xy-coordinate system is a rectangular coordinate system, it is possible to drop perpendiculars from M and N to the x-axis and to the y-axis. Call the points of intersection E, F, G, and H.



∠MCD ≅ ∠NCD, so it follows that MD = ND. $\overleftrightarrow{NF} \parallel \overleftrightarrow{ME} \parallel y-axis$ and $\overleftrightarrow{MG} \parallel \overleftrightarrow{NH} \parallel x-axis$. Also, the *x*-axis is perpendicular to the *y*-axis. So $\overleftrightarrow{ME} \perp x-axis$, $\overleftrightarrow{NF} \perp x-axis$, $\overleftrightarrow{MG} \perp y-axis$, and $\overleftrightarrow{NH} \perp y-axis$. Therefore, ΔDEO ≅ ΔDHO, and it follows that DE = DH. ME = MD + DE = ND + DH = *a*. That is, the *y*-coordinate of M is *a*. Thus, DNFE is a rectangle, and MDHG is a rectangle. Hence, GM = DH and NF = DE. Since DE = DH, GM = NF = *b*. That is, the *x*-coordinate of M is *b*. So the reflection of (*a*, *b*) in the line *y* = *x* is (*b*, *a*).

Suppose that cosecant and sine were inverse functions. A reflection of the graph of *y* = csc(*x*) in the line *y* = *x* would be the graph of *y* = sin(*x*). The figure at the right shows, on one coordinate system, graphs of the sine function, the line given by *y* = *x*, the cosecant function, and the reflection of the cosecant function over the line *y* = *x*. Because the reflection of the graph of the cosecant function over the line *y*=*x* does not coincide with the graph of the sine function, sine and cosecant are not inverse functions.



For another argument that cosecant and sine are not inverse functions, let $f(x) = \sin x$ and $g(x) = \csc x$. Consider the values of these functions in the table below:

x	*f(x) = sin x*	*g(f(x)) = csc (sin(x))*
0	0	#DIV/0!
0.523598776	0.5	2.085829643
0.785398163	0.707106781	1.539321334
1.047197551	0.866025404	1.312749454
1.570796327	1	1.188395106

The columns for *x* and *g*(*f*(*x*)) show that *g*(*f*(*x*)) ≠ *x*, so these two functions do not satisfy the property of inverse functions.

Support for the conclusion that the functions *f* and *g* are inverses if and only if *g* consists of the ordered pairs (*f*(*x*), *x*) and *f* consists of the ordered pairs (*g*(*x*), *x*) comes from consideration of issues of domain and range. In order for *f* and *g* to be inverse functions, *f*(*g*(*x*)) = *g*(*f*(*x*)) = *x*. The range of *f* must be the domain of *g*, and the range of *g* must be the domain of *f*. This is equivalent to saying that *g* consists of the ordered pairs (*f*(*x*), *x*), and *f* consists of the ordered pairs (*g*(*x*), *x*).

Mathematical Focus 3

Although the inverse under multiplication is not the same as the inverse under function composition, the same notation, the superscript -1, is used for both.

The general function notation, *y* = *f*(*x*), means that *y* is the image of *x* under the function *f*. To indicate the inverse function, the notation $x = f^{-1}(y)$ is used; it means that *x* is the image of *y* under the inverse of *f*. The superscript -1 is used to show that the inverse f^{-1} is a function related to f; the superscript is not to be interpreted as an exponent. In contrast, $z = xy$ means that z is the product of x and y, and to indicate the inverse of the product, the notation $x = y^{-1}z$ is used, where the superscript is interpreted as an exponent. These two usages of the superscript -1 are distinct.

The functions secant, cosecant, and cotangent are defined, respectively, as follows: $\sec x \cong \frac{1}{\cos x}$, $\csc x \cong \frac{1}{\sin x}$, and $\cot x \cong \frac{1}{\tan x}$. They are not defined for $\cos x = 0$, $\sin x = 0$, or $\tan x = 0$. They can be written as $\sec x \cong (\cos x)^{-1}$, $\csc x \cong (\sin x)^{-1}$, and $\cot x \cong (\tan x)^{-1}$.

In contrast, the inverse functions of sine, cosine, and tangent are, respectively, $\sin^{-1} x$ (sometimes written arcsin x), $\cos^{-1} x$ (sometimes arccos x), and $\tan^{-1} x$ (sometimes arctan x or arctg x).

These inverse trigonometric functions are multivalued. For example, there are multiple values of x such that $y = \sin x$, so $\sin^{-1} x$ is not uniquely defined and is therefore not a function unless a principal value is defined by restricting the domain. For example, $\sin^{-1} x$ is typically defined only for $-1 \le x \le 1$. Principal values are sometimes denoted with a capital letter; for example, the principal value of the inverse sine, $\sin^{-1} x$, may be denoted $Sin^{-1}x$ or Arcsin x (but this capitalization notation is far from universal and may, in fact, be used with the opposite meaning).

Post-Commentary

The trigonometric functions are traditionally defined first in terms of the ratios of the sides of a right triangle, and the three sides of the triangle give rise to six possible ratios. Because calculation with the ratios was so difficult in the days before calculators, names were given to each one and tables constructed for them. The advent of computers has meant that given one of the functions, the others are easily calculated (using trigonometric identities). The secant, cosecant, and cotangent, which were never used much in applications, have consequently diminished somewhat in importance relative to the other three.

The inverse trigonometric functions—especially the inverse sine, inverse tangent, and inverse secant—turn out to be useful in calculus as antiderivatives for integrals involving quotients and roots of polynomials. They are often used in precalculus courses primarily to illustrate the concept of inverse function.